p-Adic Numbers and Generalization of Number Concept

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Contents

Abstract

In this chapter the general TGD inspired mathematical ideas related to p-adic numbers are discussed. The extensions of the p-adic numbers including extensions containing transcendentals, the correspondences between p-adic and real numbers, p-adic differential and integral calculus, and p-adic symmetries and Fourier analysis belong the topics of the chapter.

The basic hypothesis is that p-adic space-time regions correspond to cognitive representations for the real physics appearing already at the elementary particle level. The interpretation of the p-adic physics as a physics of cognition is justified by the inherent p-adic non-determinism of the p-adic differential equations making possible the extreme flexibility of imagination.

p-Adic canonical identification and the identification of reals and p-adics by common rationals are the two basic identification maps between p-adics and reals and can be interpreted as two basic types of cognitive maps. The concept of p-adic fractality is defined and p-adic fractality is the basic property of the cognitive maps mapping real world to the p-adic internal world. Canonical identification is not general coordinate invariant and at the fundamental level it is applied only to map p-adic probabilities and predictions of p-adic thermodynamics to real numbers. The correspondence via common rationals is general coordinate invariant correspondence when general coordinate transformations are restricted to rational or extended rational maps: this has interpretation in terms of fundamental length scale unit provided by CP_2 length.

A natural outcome is the generalization of the notion of number. Different number fields form a book like structure with number fields and their extensions representing the pages of the book glued together along common rationals representing the rim of the book. This generalization forces also the generalization of the manifold concept: both imbedding space and configuration space are obtained as union of copies corresponding to various number fields glued together along common points, in particular rational ones. Space-time surfaces decompose naturally to real and p-adic space-time sheets. In this framework the fusion of real and various p-adic physics reduces more or less to to an algebraic continuation of rational number based physics to various number fields and their extensions.

p-Adic differential calculus obeys the same rules as real one and an interesting outcome are p-adic fractals involving canonical identification. Perhaps the most crucial ingredient concerning the practical formulation of the p-adic physics is the concept of the p-adic valued definite integral. Quite generally, all general coordinate invariant definitions are based on algebraic continuation by common rationals. Integral functions can be defined using just the rules of ordinary calculus

and the ordering of the integration limits is provided by the correspondence via common rationals. Residy calculus generalizes to p-adic context and also free Gaussian functional integral generalizes to p-adic context and is expected to play key role in quantum TGD at configuration space level.

The special features of p-adic Lie-groups are briefly discussed: the most important of them being an infinite fractal hierarchy of nested groups. Various versions of the p-adic Fourier analysis are proposed: ordinary Fourier analysis generalizes naturally only if finite-dimensional extensions of p-adic numbers are allowed and this has interpretation in terms of p-adic length scale cutoff. Also p-adic Fourier analysis provides a possible definition of the definite integral in the p-adic context by using algebraic continuation.

1 Introduction

There have been a lot of early speculations about the role of the p-adic numbers in Physics
[16, 17, 18]. In [19] one can find a review of the work done. In general the work is related to the quantum theory and based on the assumption that the quantum mechanical state space is an ordinary complex Hilbert space. This is not absolutely necessary since p-adic unitarity and probability concepts make sense [20]. What is however essential is some kind of correspondence between the p-adic and real numbers since the predictions of, say p-adic quantum mechanics, should be expressed in terms of the real numbers.

One can imagine two kinds of correspondences between reals and p-adics.

a) The correspondence defined by rational numbers regarded as common to real and p-adic number fields and their extensions applies at the level of geometry. The generalization of the number concept obtained by gluing all number fields together along common rational numbers generalizes also to the level of manifolds and Hilbert spaces.

b) Another correspondence is based on the canonical identification and can be used to map p-adic probabilities to their real counterparts. Also the predictions of p-adic thermodynamics for mass squared values of elementary particles can be mapped to the p-adic numbers using the correspondence. Canonical identification does not however work at space-time level since it does not respect field equations nor even differentiability although it is continuous.

c) A compromize between canonical correspondence and identification via common rationals is achieved by using a modification of canonical iden-

tification $I_{R_p \to R}$ defined as $I_1(r/s) = I(r)/I(s)$.

The formulation of the p-adic physics requires the construction of the p-adic differential and integral calculus. Also the p-adic counterparts of Hilbert space, group theory, and Fourier analysis are needed as also the generalization of manifold concept, Riemann geometry, sub-manifold geometry, and even configuration space geometry. These generalizations are discussed in this and subsequent chapter.

1.1 Canonical identification

The notion of canonical identification dominated p-adic TGD almost for a decade. Canonical identification is a canonical correspondence between the p-adic numbers and nonnegative real numbers defined by the "pinary" expansion of real number: positive real number $x = \sum x_n p^n$ ($x = 0, 1, .., p-1$, p prime) is mapped to p-adic number $\sum x_n p^{-n}$. This canonical correspondence allows to induce p-adic topology to the real axis. p-Adically differentiable functions define typically fractal like real functions via the canonical identification so that p-adic numbers provide analytic tool for producing fractals. p-Adic numbers allow algebraic extensions of arbitrary dimension and the concept of complex analyticity generalizes to p-adic analyticity.

The concepts of the p-adic probability and unitarity make sense and one can associate with the p-adic probabilities unique real probabilities using the canonical correspondence and this predicts novel physical effects. The successful p-adic description of the particle massivation relies heavily on the canonical correspondence.

1.2 Identification via common rationals

Besides canonical identification there is also a second natural correspondence between reals and p-adics. This correspondence is induced via common rationals in the sense that one can regard p-adics and reals as different completions of rationals and given rational number can be identified as an element or reals or of any p-adic number field.

For instance, if S-matrix is complex rational matrix or belongs to finite-dimensional extension or rationals, one can regard it as either real or p-adic S-matrix. The assumption that the so called CKM matrix describing quark mixings is complex rational, fixes with some empirical inputs the CKM matrix essentially uniquely. Second example: if it is assumed that fundamental state space has complex rationals as a coefficient field, it becomes sensible to define tensor factors of Hilbert spaces belonging to different number fields

because entanglement is possible with complex rational coefficients. One could also see the basic physics as essentially rational and real and p-adic physics as different algebraic continuations of it. Also much more general vision encouraged by TGD inspired theory of consciousness and p-adic physics as physics of cognition and intentionality is possible.

On can generalize the concepts of the definite integral, Hilbert space, Riemannian manifold, and Lie group to the p-adic context in a relatively straightforward manner and the correspondence via common rationals makes it possible to carry out these generalizations as an algebraic continuation with clear interpretation about what is involved. The generalization of the number concept generalizes these structures so that real and various p-adic variants of the structure can be seen as various facets of the generalized structure.

1.3 Hybrid of canonical identification and identification via common rationals

A compromize between canonical correspondence and identification via common rationals is achieved by using a modification of canonical identification $I_{R_p \to R}$ defined as $I_1(r/s) = I(r)/I(s)$. If the conditions $r \ll p$ and $s \ll p$ hold true, the map respects algebraic operations and also unitarity and various symmetries. It seems that this option must be used to relate p-adic transition amplitudes to real ones and vice versa [F5]. In particular, real and p-adic coupling constants are related by this map. Also some problems related to p-adic mass calculations find a nice resolution when I_1 is used.

This variant of canonical identification is not equivalent with the original one using the infinite expansion of q in powers of p since canonical identification does not commute with product and division. The variant is however unique in the recent context when r and s in $q = r/s$ have no common factors. For integers $n < p$ it reduces to direct correspondence. R_{p1} and R_{p_2} are glued together along common rationals by an the composite map $I_{R \to R_{p_2}} I_{R_{p_1} \to R}$.

1.4 Topics of the chapter

The topics of the chapter are the following:

a) p-Adic numbers, their extensions (also those involving transcendentals) are described. The existence of a square root of an ordinary p-adic number is necessary in many applications of the p-adic numbers (p-adic group theory, p-adic unitarity, Riemannian geometry) and its existence im-

plies a unique algebraic extension, which is 4-dimensional for $p > 2$ and 8-dimensional for $p = 2$. Contrary to the first expectations, all possible algebraic extensions are possible and one cannot interpret the algebraic dimension of the algebraic extension as a physical dimension.

b) The concepts of the p-adic differentiability and analyticity are discussed. The notion of p-adic fractal is introduced the properties of the fractals defined by p-adically differentiable functions are discussed.

c) Various approaches to the problem of defining p-adic valued definite integral are discussed. The only reasonable generalizations rely on algebraic continuation and correspondence via common rationals. p-Adic field equations do not necessitate p-adic definite integral but algebraic continuation allows to assign to a given real space-time sheets a p-adic space-time sheets if the definition of space-time sheet involves algebraic relations between imbedding space coordinates. There are also hopes that one can algebraically continue the value of Kähler action to p-adic context if finite-dimensional extensions are allowed.

d) Symmetries are discussed from p-adic point of view starting from the identification via common rationals. Also possible p-adic generalizations of Fourier analysis are considered. Besides a number theoretical approach, group theoretical approach providing a direct generalization of the ordinary Fourier analysis based on the utilization of exponent functions existing in algebraic extensions containing some root of e and its powers up to e^{p-1} is discussed. Also the generalization of Fourier analysis based on the Pythagorean phases is considered.

2   p-Adic numbers

2.1   Basic properties of p-adic numbers

p-Adic numbers (p is prime: 2,3,5,...) can be regarded as a completion of the rational numbers using a norm, which is different from the ordinary norm of real numbers [21]. p-Adic numbers are representable as power expansion of the prime number p of form:

$$ x \;\; = \;\; \sum_{k \geq k_0} x(k)p^k, \;\; x(k) = 0, ..., p-1 \;\; . \tag{1} $$

The norm of a p-adic number is given by

$$|x| = p^{-k_0(x)} \quad . \tag{2}$$

Here $k_0(x)$ is the lowest power in the expansion of the p-adic number. The norm differs drastically from the norm of the ordinary real numbers since it depends on the lowest pinary digit of the p-adic number only. Arbitrarily high powers in the expansion are possible since the norm of the p-adic number is finite also for numbers, which are infinite with respect to the ordinary norm. A convenient representation for p-adic numbers is in the form

$$x = p^{k_0} \varepsilon(x) \quad , \tag{3}$$

where $\varepsilon(x) = k + ...$ with $0 < k < p$, is p-adic number with unit norm and analogous to the phase factor $exp(i\phi)$ of a complex number.

The distance function $d(x, y) = |x - y|_p$ defined by the p-adic norm possesses a very general property called ultra-metricity:

$$d(x, z) \leq max\{d(x, y), d(y, z)\} \quad . \tag{4}$$

The properties of the distance function make it possible to decompose R_p into a union of disjoint sets using the criterion that x and y belong to same class if the distance between x and y satisfies the condition

$$d(x, y) \leq D \quad . \tag{5}$$

This division of the metric space into classes has following properties:

a) Distances between the members of two different classes X and Y do not depend on the choice of points x and y inside classes. One can therefore speak about distance function between classes.

b) Distances of points x and y inside single class are smaller than distances between different classes.

c) Classes form a hierarchical tree.

Notice that the concept of the ultra-metricity emerged in physics from the models for spin glasses and is believed to have also applications in biology [25]. The emergence of p-adic topology as the topology of the effective space-time would make ultra-metricity property basic feature of physics.

2.2 p-Adic ultrametricity and divergence cancellation

p-Adic ultrametricity implies that the p-adic norm for a sum of p-adic numbers cannot be larger than the maximum of the p-adic norm for the summands. In p-adic QFT this has an overall important consequence: p-adic loops sums over the discrete labels characterizing p-adic planewaves are bounded from above. This means an automatic cancellation of the ultraviolet divergences. The finite volume of the p-adic space-time region in turn implies the cancellation of the infrared divergences and the convergence of the p-adic loops sums to a well defined limit.

It must be emphasized that the finiteness of the terms appearing in the loop sums is not trivially true in the coordinate-space formulation of the perturbation theory and it will be found that finiteness, or equivalently, the p-adic pseudo-constancy of the coordinate space propagators, might necessitate the natural p-adic cutoff provided by the CP_2 radius below which the assumption about the effective quantum average space-time representable locally as a map $M_+^4 \to CP_2$ fails. One must however emphasize that the formulation of the theory is not yet so detailed that one could draw any strong conclusions in this respect.

2.3 Extensions of p-adic numbers

Algebraic democracy suggests that all possible real algebraic extensions of the p-adic numbers are possible. This conclusion is also suggested by various physical requirements, say the fact that the eigenvalues of a Hamiltonian representable as a rational or p-adic $N \times N$-matrix, being roots of N:th order polynomial equation, in general belong to an algebraic extension of rationals or p-adics. The dimension of the algebraic extension cannot be interpreted as physical dimension. Algebraic extensions are characteristic for cognitive physics and provide a new manner to code information. A possible interpretation for the algebraic dimension is as a dimension for a cognitive representation of space and would explain how it is possible to mathematically imagine spaces with all possible dimensions although physical space-time dimension is four (TGD as a number theory vision suggest that also space-time dimensions which are multiples of four are possible). The idea of algebraic hologram and other ideas related to the physical interpretation of the algebraic extensions of p-adics are discussed in the chapter "TGD as a generalized number theory".

It seems however that algebraic democracy must be extended to include also transcendentals in the sense that finite-dimensional extensions involving

also transcendental numbers are possible: for instance, Neper number e defines a p-dimensional extension. It has become clear that these extensions fundamental for understanding how p-adic physics as physics of cognition is able to mimick real physics. The evolution of mathematical cognition can be seen as a process in which p-adic space-time sheets involving increasing value of p-adic prime p and increasing dimension of algebraic extension appear in quantum jumps.

2.3.1 Recipe for constructing algebraic extensions

Real numbers allow only complex numbers as an algebraic extension. For p-adic numbers algebraic extensions of arbitrary dimension are possible [21]. The simplest manner to construct (n+1)-dimensional extensions is to consider irreducible polynomials $P_n(t)$ in R_p assumed to have rational coefficients: irreduciblity means that the polynomial does not possess roots in R_p so that one cannot decompose it into a product of lower order R_p valued polynomials. This condition is equivalent with the condition with irreducibility in the finite field $G(p, 1)$, that is modulo p in R_p.

Denoting one of the roots of $P_n(t)$ by θ and defining $\theta^0 = 1$ the general form of the extension is given by

$$Z = \sum_{k=0,..,n-1} x_k \theta^k \ . \tag{6}$$

Since θ is root of the polynomial in R_p it follows that θ^n is expressible as a sum of lower powers of θ so that these numbers indeed form an n-dimensional linear space with respect to the p-adic topology.

Especially simple odd-dimensional extensions are cyclic extensions obtained by considering the roots of the polynomial

$$
\begin{aligned}
P_n(t) &= t^n + \epsilon d \ , \\
\epsilon &= \pm 1 \ .
\end{aligned}
\tag{7}
$$

For $n = 2m+1$ and $(n = 2m, \epsilon = +1)$ the irreducibility of $P_n(t)$ is guaranteed if d does not possess n:th root in R_p. For $(n = 2m, \epsilon = -1)$ one must assume that $d^{1/2}$ does not exist p-adically. In this case θ is one of the roots of the equation

$$t^n = \pm d \ , \tag{8}$$

where d is a p-adic integer with a finite number of pinary digits. It is possible although not necessary to identify the roots as complex numbers. There exists n complex roots of d and θ can be chosen to be one of the real or complex roots satisfying the condition $\theta^n = \pm d$. The roots can be written in the general form

$$
\begin{aligned}
\theta &= d^{1/n} exp(i\phi(m)), \ m = 0, 1, ..., n-1 \ , \\
\phi(m) &= \frac{m2\pi}{n} \ \text{or} \ \frac{m\pi}{n} \ .
\end{aligned}
\tag{9}
$$

Here $d^{1/n}$ denotes the real root of the equation $\theta^n = d$. Each of the phase factors $\phi(m)$ gives rise to an algebraically equivalent extension: only the representation is different. Physically these extensions need not be equivalent since the identification of the algebraically extended p-adic numbers with the complex numbers plays a fundamental role in the applications. The cases $\theta^n = \pm d$ are physically and mathematically quite different.

2.3.2 p-Adic valued norm for numbers in algebraic extension

The p-adic valued norm of an algebraically extended p-adic number x can be defined as some power of the ordinary p-adic norm of the determinant of the linear map $x :^e R_p^n \to^e R_p^n$ defined by the multiplication with x: $y \to xy$

$$
N(x) = det(x)^\alpha \ , \ \alpha > 0 \ .
\tag{10}
$$

Real valued norm can be defined as the p-adic norm of $N(x)$. The requirement that the norm is homogenous function of degree one in the components of the algebraically extended 2-adic number (like also the standard norm of R^n) implies the condition $\alpha = 1/n$, where n is the dimension of the algebraic extension.

The canonical correspondence between the points of R_+ and R_p generalizes in obvious manner: the point $\sum_k x_k \theta^k$ of algebraic extension is identified as the point $(x_R^0, x_R^1, ..., x_R^k, ..)$ of R^n using the pinary expansions of the components of p-adic number. The p-adic linear structure of the algebraic extension induces a linear structure in R_+^n and p-adic multiplication induces a multiplication for the vectors of R_+^n.

2.3.3 Algebraic extension allowing square root of ordinary p-adic numbers

The existence of a square root of an ordinary p-adic number is a common theme in various applications of the p-adic numbers and for long time I erratically believed that only this extension is involved with p-adic physics. Despite this square root allowing extension is of central importance and deserves a more detailed discussion.

a) The p-adic generalization of the representation theory of the ordinary groups and Super Kac Moody and Super Virasoro algebras exists provided an extension of the p-adic numbers allowing square roots of the 'real' p-adic numbers is used. The reason is that the matrix elements of the raising and lowering operators in Lie-algebras as well as of oscillator operators typically involve square roots. The existence of square root might play a key role in various p-adic considerations.

b) The existence of a square root of a real p-adic number is also a necessary ingredient in the definition of the p-adic unitarity and probability concepts since the solution of the requirement that $p_{mn} = S_{mn}\bar{S}_{mn}$ is ordinary p-adic number leads to expressions involving square roots.

c) p-Adic length scales hypothesis states that the p-adic length scale is proportional to the square root of p-adic prime.

d) Simple metric geometry of polygons involves square roots basically via the theorem of Pythagoras. p-Adic Riemannian geometry necessitates the existence of square root since the definition of the infinitesimal length ds involves square root. Note however that p-adic Riemannian geometry can be formulated as a mere differential geometry without any reference to global concepts like lengths, areas, or volumes.

The original belief that square root allowing extensions of p-adic numbers are exceptional seems to be wrong in light of TGD as a generalized number theory vision. All algebraic extensions of p-adic numbers a possible and the interpretation of algebraic dimension of the extension as a physical dimension is not the correct thing to do. Rather, the possibility of arbitrarily high algebraic dimension reflects the ability of mathematical cognition to imagine higher-dimensional spaces. Square root allowing extension of the p-adic numbers is the simplest one imaginable, and it is fascinating that it indeed is the dimension of space-time surface for $p > 2$ and dimension of imbedding space for $p = 2$. Thus the square root allowing extensions deserve to be discussed.

The results can be summarized as follows.

a) In $p > 2$ case the general form of extension is

$$Z \quad = \quad (x + \theta y) + \sqrt{p}(u + \theta v) \ , \tag{11}$$

where the condition $\theta^2 = x$ for some p-adic number x not allowing square root as a p-adic number. For $p \ mod \ 4 = 3 \ \theta$ can be taken to be imaginary unit. This extension is natural for p-adication of space-time surface so that space-time can be regarded as a number field locally. Imbedding space can be regarded as a cartesian product of two 4-dimensional extensions locally.

b) In $p = 2$ case 8-dimensional extension is needed to define square roots. The extension is defined by adding $\theta_1 = \sqrt{-1} \equiv i$, $\theta_2 = \sqrt{2}$, $\theta_3 = \sqrt{3}$ and the products of these so that the extension can be written in the form

$$Z \quad = \quad x_0 + \sum_k x_k \theta_k + \sum_{k<l} x_{kl} \theta_{kl} + x_{123} \theta_1 \theta_2 \theta_3 \ . \tag{12}$$

Clearly, $p = 2$ case is exceptional as far as the construction of the conformal field theory limit is considered since the structure of the representations of Virasoro algebra and groups in general changes drastically in $p = 2$ case. The result suggest that in $p = 2$ limit space-time surface and H are in same relation as real numbers and complex numbers: space-time surfaces defined as the absolute minima of 2-adiced Kähler action are perhaps identifiable as surfaces for which the imaginary part of 2-adically analytic function in H vanishes.

The physically interesting feature of p-adic group representations is that if one doesn't use $\sqrt{p}$ in the extension the number of allowed spins for representations of $SU(2)$ is finite: only spins $j < p$ are allowed. In $p = 3$ case just the spins $j \leq 2$ are possible. If 4-dimensional extension is used for $p = 2$ rather than 8-dimensional then one gets the same restriction for allowed spins.

2.3.4 Finite-dimensional extensions involving transcendentals

The transcendentals e and π appear repeatedly in the basic formulas of calculus and physics. Also logarithms are unavoidable. The idea that rational numbers are common for all number fields suggests that the p-adic variant of logarithm function should be well-defined and be equivalent with the real logarithm in the subset of rationals. This boils down to the requirement that the logarithms $log(p)$, p prime exist for all primes.

The requirement that cognition has as its space-time correlates p-adic space-time sheets corresponding to finite-dimensional extensions of p-adic

numbers implies that the extensions involving transcendentals must be finite-dimensional. This requirement discussed in the chapter "Riemann Hypothesis and Physics" looks extremely strong.

The intuitive expectation is that the extension containing e, π, and logarithms $log(p)$ of primes is finite-dimensional for any prime p. Log(p) is contained in the extension if $\pi/log(p)$ is rational number for any prime p. π is contained in the extension of $\pi/log(log(....(log(\pi)...))$ is rational number for sum finite-fold logarithmic iterate of π. The detailed argument is discussed in the chapter "Riemann Hypothesis and Physics" and here only a rough sketch is given.

a) The extension containing e is finite-dimensional. The reason is that e^x exists as a p-adic series for $|x|_p < 1$. Thus only the powers $e, e^2, ..., e^{p-1}$ need to be introduced and this gives to a p-dimensional extension.

b) One might think that π can be defined in the extension containing $\sqrt{-1}$ ($\sqrt{-1}$ is an ordinary p-adic number for $p \ mod \ 4 = 1$) by using the identity $log(-1) = \sqrt{-1}\pi$ and by writing $log(-1) = log[(p-1)/(1-p)] = 1/2log[(p-1)^2] - log(1-p)$ and by applying power series of logarithm $log(1+y)$ converging for $|y|_p < 1$. Unfortunately, the constraint $exp(i\pi) = -1$ is not satisfied for this identification of π. Thus the only hope is that e/π is rational number or an analogous statement holds true for some higher logarithmic iterate of π.

c) The logarithms $log(q)$, $q \neq p$, can be defined by writing

$$log(q) = log[q^{d(p,q)}]/d(p,q) \ ,$$

where $d(p,q)$ is an integer such that $q^{d(p,q)} \ mod \ p = 1$. The difficult part is thus the identification of $log(p)$ for R_p. This logarithm exists if $log(p)/\pi$ is a rational number. This number theoretical conjecture is unproven and implies that $log(x)/\pi$ is rational number for any rational number x. The conjecture follows from the requirement that Riemann Zeta is a universal function existing in the field of real numbers and in various p-adic number fields and is algebraically continuable from its representation in the set of rationals. This is achieved if the values of the functions p^{iy} appearing as building blocks of Riemann Zeta $\zeta(x+iy)$ are algebraic numbers when y is a rational number. A stronger condition is that y is rational number for the zeros $z = 1/2 + iy$ of Riemann zeta so that also zeros would be universal.

2.4   p-Adic Numbers and Finite Fields

Finite fields (Galois fields) consists of finite number of elements and allow sum, multiplication and division. A convenient representation for the ele-

ments of a finite field is as the roots of the polynomial equation $t^{p^m} - t = 0 \ mod \ p$, where p is prime, m an arbitrary integer and t is element of a field of characteristic p ($pt = 0$ for each t). The number of elements in a finite field is p^m, that is power of prime number and the multiplicative group of a finite field is group of order $p^m - 1$. $G(p, 1)$ is just cyclic group Z_p with respect to addition and $G(p, m)$ is in rough sense m:th Cartesian power of $G(p, 1)$.

The elements of the finite field $G(p, 1)$ can be identified as the p-adic numbers $0, ..., p-1$ with p-adic arithmetics replaced with modulo p arithmetics. The finite fields $G(p, m)$ can be obtained from m-dimensional algebraic extensions of the p-adic numbers by replacing the p-adic arithmetics with the modulo p arithmetics. In TGD context only the finite fields $G(p > 2, 2)$, $p \ mod \ 4 = 3$ and $G(p = 2, 4)$ appear naturally. For $p > 2$, $p \ mod \ 4 = 3$ one has: $x + iy + \sqrt{p}(u + iv) \rightarrow x_0 + iy_0 \in G(p, 2)$.

An interesting observation is that the unitary representations of the p-adic scalings $x \rightarrow p^k x \ k \in Z$ lead naturally to finite field structures. These representations reduce to representations of a finite cyclic group Z_m if $x \rightarrow p^m x$ acts trivially on representation functions for some value of m, $m = 1, 2, ...$ Representation functions, or equivalently the scaling momenta $k = 0, 1, ..., m-1$ labelling them, have a structure of cyclic group. If $m \neq p$ is prime the scaling momenta form finite field $G(m, 1) = Z_m$ with respect to the summation and multiplication modulo m. Also the p-adic counterparts of the ordinary plane waves carrying p-adic momenta $k = 0, 1..., p-1$ can be given the structure of Finite Field $G(p, 1)$: one can also define complexified plane waves as square roots of the real p-adic plane waves to obtain Finite Field $G(p, 2)$.

3 What is the correspondence between p-adic and real numbers?

There must be some kind of correspondence between reals and p-adic numbers. This correspondence can depend on context. In p-adic mass calculations one must map p-adic mass squared values to real numbers in a continuous manner and canonical identification is a natural guess. Presumably also p-adic probabilities should be mapped to their real counterparts. One can wonder whether p-adic valued S-matrix has any physical meaning or whether one should assume that the elements of S-matrix are algebraic numbers allowing interpretation as real or p-adic numbers: this would pose extremely strong constraints on S-matrix. If one wants to introduce p-adic

physics at space-time level one must be able to relate p-adic and real space-time regions to each other and the identification along common rational points of real and various p-adic variants of the imbedding space suggests itself here.

3.1 Generalization of the number concept

The recent view about the unification of real and p-adic physics is based on the generalization of number concept obtained by fusing together real and p-adic number fields along common rationals.

3.1.1 Rational numbers as numbers common to all number fields

The unification of real physics of material work and p-adic physics of cognition and intentionality leads to the generalization of the notion of number field. Reals and various p-adic number fields are glued along their common rationals (and common algebraic numbers too) to form a fractal book like structure. Allowing all possible finite-dimensional extensions of p-adic numbers brings additional pages to this "Big Book".

This generalization leads to a generalization of the notion of manifold as a collection of a real manifold and its p-adic variants glued together along common rationals. The precise formulation involves of course several technical problems. For instance, should one glue along common algebraic numbers and Should one glue along common transcendentals such as e^p? Are algebraic extensions of p-adic number fields glued together along the algebraics too?

At space-time level the book like structure corresponds to the decomposition of space-time surface to real and p-adic space-time sheets. This has deep implications for the view about cognition. For instance, two points infinitesimally near p-adically are infinitely distant in real sense so that cognition becomes a cosmic phenomenon.

3.1.2 How large p-adic space-time sheets can be?

Space-time region having finite size in the real sense can have arbitrarily large size in p-adic sense and vice versa. This raises a rather thought provoking questions. Could the p-adic space-time sheets have cosmological or even infinite size with respect to the real metric but have be p-adically finite? How large space-time surface is responsible for the p-adic representation of my body? Could the large or even infinite size of the cognitive space-time sheets explain why creatures of a finite physical size can invent

the notion of infinity and construct cosmological theories? Could it be that pinary cutoff $O(p^n)$ defining the resolution of a p-adic cognitive representation would define the size of the space-time region needed to realize the cognitive representation?

In fact, the mere requirement that the neighborhood of a point of the p-adic space-time sheet contains points, which are p-adically infinitesimally near to it can mean that points infinitely distant from this point in the real sense are involved. A good example is provided by an integer valued point $x = n < p$ and the point $y = x + p^m$, $m > 0$: the p-adic distance of these points is p^{-m} whereas at the limit $m \to \infty$ the real distance goes as p^m and becomes infinite for infinitesimally near points. The points $n + y$, $y = \sum_{k>0} x_k p^k$, $0 < n < p$, form a p-adically continuous set around $x = n$. In the real topology this point set is discrete set with a minimum distance $\Delta x = p$ between neighboring points whereas in the p-adic topology every point has arbitrary nearby points. There are also rationals, which are arbitrarily near to each other both p-adically and in the real sense. Consider points $x = m/n$, m and n not divisible by p, and $y = (m/n) \times (1 + p^k r)/(1 + p^k s)$, $s = r + 1$ such that neither r or s is divisible by p and $k >> 1$ and $r >> p$. The p-adic and real distances are $|x-y|_p = p^{-k}$ and $|x-y| \simeq (m/n)/(r+1)$ respectively. By choosing k and r large enough the points can be made arbitrarily close to each other both in the real and p-adic senses.

The idea about astrophysical size of the p-adic cognitive space-time sheets providing representation of body and brain is consistent with TGD inspired theory of consciousness, which forces to take very seriously the idea that even human consciousness involves astrophysical length scales.

3.1.3 Generalizing complex analysis by replacing complex numbers by generalized numbers

One general idea which results as an outcome of the generalized notion of number is the idea of a universal function continuable from a function mapping rationals to rationals or to a finite extension of rationals to a function in any number field. This algebraic continuation is analogous to the analytical continuation of a real analytic function to the complex plane. Rational functions with rational coefficients are obviously functions satisfying this constraint. Algebraic functions with rational coefficients satisfy this requirement if appropriate finite-dimensional algebraic extensions of p-adic numbers are allowed. Exponent function is such a function. Logarithm is also such a function provided that the above mentioned number theoretic conjecture holds true.

For instance, residy calculus might be generalized so that the value of an integral along the real axis could be calculated by continuing it instead of the complex plane to any number field via its values in the subset of rational numbers forming the rim of the book like structure having number fields as its pages. If the poles of the continued function in the finitely extended number field allow interpretation as real numbers it might be possible to generalize the residy formula. One can also imagine of extending residy calculus to any algebraic extension. An interesting situation arises when the poles correspond to extended p-adic rationals common to different pages of the "great book". Could this mean that the integral could be calculated at any page having the pole common. In particular, could a p-adic residy integral be calculated in the ordinary complex plane by utilizing the fact that in this case numerical approach makes sense.

3.2 Canonical identification

There exists a natural continuous map $Id : R_p \to R_+$ from p-adic numbers to non-negative real numbers given by the "pinary" expansion of the real number for $x \in R$ and $y \in R_p$ this correspondence reads

$$
\begin{aligned}
y &= \sum_{k>N} y_k p^k \to x = \sum_{k<N} y_k p^{-k} \ , \\
y_k &\in \{0, 1, .., p-1\} \ .
\end{aligned}
\tag{13}
$$

This map is continuous as one easily finds out. There is however a little difficulty associated with the definition of the inverse map since the pinary expansion like also desimal expansion is not unique $(1 = 0.999...)$ for the real numbers x, which allow pinary expansion with finite number of pinary digits

$$
\begin{aligned}
x &= \sum_{k=N_0}^{N} x_k p^{-k} \ , \\
x &= \sum_{k=N_0}^{N-1} x_k p^{-k} + (x_N - 1)p^{-N} + (p-1)p^{-N-1} \sum_{k=0,..} p^{-k} \ .
\end{aligned}
\tag{14}
$$

The p-adic images associated with these expansions are different

$$
\begin{aligned}
y_1 &= \sum_{k=N_0}^{N} x_k p^k \ , \\
y_2 &= \sum_{k=N_0}^{N-1} x_k p^k + (x_N - 1)p^N + (p-1)p^{N+1} \sum_{k=0,..} p^k \\
&= y_1 + (x_N - 1)p^N - p^{N+1} \ , \tag{15}
\end{aligned}
$$

so that the inverse map is either two-valued for p-adic numbers having expansion with finite number of pinary digits or single valued and discontinuous and nonsurjective if one makes pinary expansion unique by choosing the one with finite number of pinary digits. The finite number of pinary digits expansion is a natural choice since in the numerical work one always must use a pinary cutoff on the real axis.

3.2.1 Canonical identification is continuous map of non-negative reals to p-adics

The topology induced by the canonical identification map in the set of positive real numbers differs from the ordinary topology. The difference is easily understood by interpreting the p-adic norm as a norm in the set of the real numbers. The norm is constant in each interval $[p^k, p^{k+1})$ (see Fig. 3.2.1) and is equal to the usual real norm at the points $x = p^k$: the usual linear norm is replaced with a piecewise constant norm. This means that p-adic topology is coarser than the usual real topology and the higher the value of p is, the coarser the resulting topology is above a given length scale. This hierarchical ordering of the p-adic topologies will be a central feature as far as the proposed applications of the p-adic numbers are considered.

Ordinary continuity implies p-adic continuity since the norm induced from the p-adic topology is rougher than the ordinary norm. This allows two alternative interpretations. Either p-adic image of a physical systems provides a good representation of the system above some pinary cutoff or the physical system can be genuinely p-adic below certain length scale L_p and become in good approximation real, when a length scale resolution L_p is used in its description. The first interpretation is correct if canonical identification is interpreted as a cognitive map. p-Adic continuity implies ordinary continuity from right as is clear already from the properties of the p-adic norm (the graph of the norm is indeed continuous from right). This feature is one clear signature of the p-adic topology.
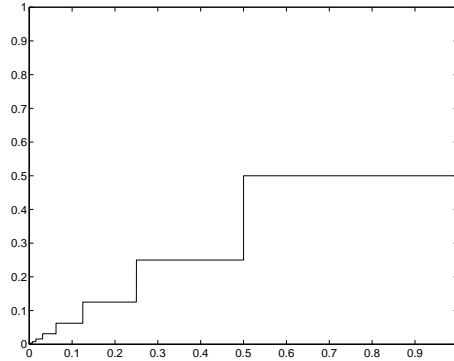
Figure 1: The real norm induced by canonical identification from 2-adic norm.

If one considers seriously the application of canonical identification to basic quantum TGD one cannot avoid the question about the p-adic counterparts of the negative real numbers. It has turned out that there is no satisfactory manner to circumvent the fact that canonical images of p-adic numbers are naturally non-negative. The correct conclusion is that canonical interpretation applies only in p-adic thermodynamics, where it is used only in the direction $R_p \to R$ and real images are naturally non-negative numbers.

3.2.2 The notion of p-adic linearity

The linear structure of the p-adic numbers induces a corresponding structure in the set of the non-negative real numbers and p-adic linearity in general differs from the ordinary concept of linearity. For example, p-adic sum is equal to real sum only provided the summands have no common pinary digits. Furthermore, the condition $x +_p y < max\{x, y\}$ holds in general for the p-adic sum of the real numbers. p-Adic multiplication is equivalent with the ordinary multiplication only provided that either of the members of the product is power of p. Moreover one has $x \times_p y < x \times y$ in general. An interesting possibility is that p-adic linearity might replace the ordinary linearity in some strongly nonlinear systems so these systems would look simple in the p-adic topology.

3.2.3 Does canonical identification define a generalized norm?

Canonical correspondence is quite essential in TGD:eish applications. Canonical identification makes it possible to define a p-adic valued definite integral. Canonical identification is in a key role in the successful predictions of the elementary particle masses. Canonical identification makes also possible to understand the connection between p-adic and real probabilities. These and many other succesfull applications suggests that canonical identification is involved with some deeper mathematical structure. The following inequalities hold true:

$$
\begin{aligned}
(x + y)_R &\leq & x_R + y_R \;, \\
|x|_p |y|_R \leq (xy)_R &\leq & x_R y_R \;,
\end{aligned}
\tag{16}
$$

where $|x|_p$ denotes p-adic norm. These inequalities can be generalized to the case of $(R_p)^n$ (a linear vector space over the p-adic numbers).

$$
\begin{aligned}
(x + y)_R &\leq & x_R + y_R \;, \\
|\lambda|_p |y|_R \leq (\lambda y)_R &\leq & \lambda_R y_R \;,
\end{aligned}
\tag{17}
$$

where the norm of the vector $x \in T_p^n$ is defined in some manner. The case of Euclidian space suggests the definition

$$
(x_R)^2 = (\sum_n x_n^2)_R \;.
\tag{18}
$$

These inequalities resemble those satisfied by the vector norm. The only difference is the failure of linearity in the sense that the norm of a scaled vector is not obtained by scaling the norm of the original vector. Ordinary situation prevails only if the scaling corresponds to a power of p.

These observations suggests that the concept of a normed space or Banach space might have a generalization and physically the generalization might apply to the description of some nonlinear systems. The nonlinearity would be concentrated in the nonlinear behavior of the norm under scaling.

3.3 The interpretation of canonical identification

During the development of p-adic TGD two seemingly mutually inconsistent competing identifications of reals and p-adics have caused a lot of painful

tension. Canonical identification provides one possible identification map respecting continuity whereas the identification of rationals as points common to p-adics and reals respects algebra of rationals. The resolution of the tension came from the realization that canonical identification naturally maps the predictions of p-adic probability theory and thermodynamics to real numbers. Canonical identification also maps p-adic cognitive representations to symbolic ones in the real world world or vice versa. The identification by common rationals is in turn the correspondence implied by the generalized notion of number and natural in the construction of quantum TGD proper.

3.3.1 Canonical identification maps the predictions of the p-adic probability calculus and statistical physics to real numbers

p-Adic mass calculations based on p-adic thermodynamics were the first and rather successful application of the p-adic physics (see the four chapters in [6]. The essential element of the approach was the replacement of the Boltzmann weight $e^{-E/T}$ with its p-adic generalization p^{L_0/T_p}, where L_0 is the Virasoro generator corresponding to scaling and representing essentially mass squared operator instead of energy. T_p is inverse integer valued p-adic temperature. The predicted mass squared averages were mapped to real numbers by canonical identification.

One could also construct a real variant of this approach by considering instead of the ordinary Boltzman weights the weights p^{-L_0/T_p}. The quantization of temperature to $T_p = log(p)/n$ would be a completely ad hoc assumption. In the case of real thermodynamics all particles are predicted to be light whereas in case of p-adic thermodynamics particle is light only if the ratio for the degeneracy of the lowest massive state to the degeneracy of the ground state is integer. Immense number of particles disappear from the spectrum of light particles by this criterion. For light particles the predictions are same as of p-adic thermodynamics in the lowest non-trivial order but in the next order deviations are possible.

The success of the p-adic mass calculations led to the idea that canonical identification generalizes also to the space-time level and appears even in the formulation of fundamental quantum TGD. However, when real space-time surfaces (absolute minima of Kähler action) are mapped by I^{-1} to their p-adic counterparts, one encounters several problems. The inverse of the canonical identification is two-valued; canonical identification map is not defined for negative real numbers; canonical identification is not manifestly General Coordinate Invariant concept; the direct canonical image of

the space-time surface is not p-adically differentiable. What is needed is smooth surface perhaps satisfying the p-adic counterparts of the field equations associated with the absolute minimization of the Kähler action.

Already the problems with the general covariance definitely exclude canonical identification and its variants at space-time level, and that the generalization of the number concept provides the correct approach. Even such a simple fact that canonical images are always non-negative suggests that the applications must be such that this restriction is naturally satisfied. Canonical identification can indeed be used to map the predictions of the p-adic valued statistical physics to real numbers. For instance, p-adic probabilities and the p-adic entropy can be mapped to real numbers by canonical identification. The general idea is that a faithful enough cognitive representation of the real physics can by the number theoretical constraints involved make predictions, which would be extremely difficult to deduce from real physics.

3.3.2 Canonical identification as cognitive map mapping real external world to p-adic internal world or vice versa

It is interesting to look what canonical identification does assuming that rationals are common to p-adics and reals. Canonical identication maps the rationals $q = m/n$, n not divisible by p in the range $[1, \infty]$ to the range $[0, 1]$ and vice versa. One can say that real axis is defined 'inside' $[0, 1]$ and 'outside' $[1, \infty]$ and canonical identification maps these regions to each other in a p-adically continuous manner. This suggests that canonical identification and its generalizations could provide basic building blocks for cognitive maps mapping external world to a cognitive representation inside brain. Symbolic representations of thoughts in real world would in turn involve canonical identification in the reverse sense.

The physical counterpart of the pinary cutoff is very natural. The larger the pinary cutoff p^n is, the larger the real counterpart of the p-adic image via the correspondence by common rationals is. What is small p-adically is large in real sense at the level of integers. The better the resolution of the cognitive map is, the larger the p-adic space-time sheet giving rise to the representation is. For the p-adic primes associated with elementary particles already the pinary cutoff $O(p^3) = 0$ requires macroscopic and even astrophysical length scales. The idea that our consciousness might involve astrophysical length scales via p-adic cognitive representations, is in accordance with the views forced by TGD inspired theory of consciousness but using considerations based on quite different premises [H8].

3.4 Variants of canonical identification

One can also imagine variants of canonical identification.

3.4.1 The variant of canonical identification commuting with division of integers

The basic problems of canonical identification is that it does not respect unitarity. For this reason it is not well suited for relating p-adic and real scattering amplitudes. The problem of the correspondence via direct rationals is that it does not respect continuity.

A compromize between algebra and topology is achieved by using a modification of canonical identification $I_{R_p \to R}$ defined as $I_1(r/s) = I(r)/I(s)$. If the conditions $r \ll p$ and $s \ll p$ hold true, the map respects algebraic operations and also unitarity and various symmetries. It seems that this option must be used to relate p-adic transition amplitudes to real ones and vice versa [F5]. In particular, real and p-adic coupling constants are related by this map. Also some problems related to p-adic mass calculations find a nice resolution when I_1 is used.

This variant of canonical identification is not equivalent with the original one using the infinite expansion of q in powers of p since canonical identification does not commute with product and division. The variant is however unique in the recent context when r and s in $q = r/s$ have no common factors. For integers $n < p$ it reduces to direct correspondence.

Generalized numbers would be regarded in this picture as a generalized manifold obtained by gluing different number fields together along rationals. Instead of a direct identification of real and p-adic rationals, the p-adic rationals in R_p are mapped to real rationals (or vice versa) using a variant of the canonical identification $I_{R \to R_p}$ in which the expansion of rational number $q = r/s = \sum r_n p^n / \sum s_n p^n$ is replaced with the rational number $q_1 = r_1/s_1 = \sum r_n p^{-n} / \sum s_n p^{-n}$ interpreted as a p-adic number:

$$ q \;=\; \frac{r}{s} = \frac{\sum_n r_n p^n}{\sum_m s_n p^n} \to q_1 = \frac{\sum_n r_n p^{-n}}{\sum_m s_n p^{-n}} \;\;. \tag{19} $$

R_{p1} and R_{p2} are glued together along common rationals by an the composite map $I_{R \to R_{p_2}} I_{R_{p_1} \to R}$.

This variant of canonical identification seems to be excellent candidate for mapping the predictions of p-adic mass calculations to real numbers and also for relating p-adic and real scattering amplitudes to each other [F5].

3.4.2 Phase preserving canonical identification

Before the emergence of new view about p-adic physics, the above listed problems forced to consider a modification of the canonical identification map and several options have been considered. The requirement of General Coordinate Invariance finally led to what seemed to be a unique solution to these problems. One must define canonical identification in preferred imbedding space coordinates: if preferred coordinates are not unique, the transformations between the preferred coordinates systems must commute with the modified canonical identification. Although this mapping is not relevant for the definition of fundamental theory, it might make sense if taken as a map defining cognitive representations at the level of Schröndinger amplitudes. In particular, the beautiful mathematical properties of this map and the direct connection with quantum measurement theory, suggest that one should not not keep mind open for possible applications of this map in some future theory of cognition.

The preferred coordinates are Minkowski coordinates $(m^0, m^3, m^1, m^2))$ and complex coordinates of CP_2 transforming linearly under certain Cartan sugroup $U(1) \times U(1)$ determined by the surface Y^3: these coordinates are determined modulo rotations of subgroup $SO(2) \times U(1) \times U(1)$ of Cartan subgroup of $SO(3,1) \times SU(3)$ acting as multiplication by a phase factor in case of $m^1 + im^2$ and CP_2 complex coordinates. Lorentz boosts in Cartan subgroup of $SO(3,1)$ act as multiplication by hyperbolic 'phase factor' in case of the coordinate pair $(m^0, m^3) \equiv a(cosh(\eta), sinh(\eta))$. The mapping commutes with these transformations if the phase factors are mapped as such to their p-adic counterparts, that is without canonical identification. The mapping is only possible for rational complex phase factors: they correspond to Pythagorean triangles. The coordinate $a = \sqrt{(m^0)^2 - (m^3)^2}$ and moduli of the complex coordinates are mapped using canonical identification.

Since phase preserving canonical identification is discontinuous in phase degrees of freedom, the image of the space-time surface induced by the mapping of H is in the generic case discrete and does not form a subset of any p-adic 4-surface. One can however require that p-adic space-time surface is a smooth completion of a minimal pinary cutoff of the image fixed by the requirement that p-adic counterparts of the field equations guaranteeing absolute minimization of the Kähler action are satisfied. The phenomenon of p-adic pseudo constants and nondeterminism of Kähler action give good hopes of achieving this. There is a direct connection with quantum measurement theory since the transformations of Cartan algebra commuting with the canonical identification map corresponds to a maximal set of commuting

observables in the algebra of the isometry charges.

Although it seems that phase preserving canonical identification might not be useful at the level of imbdeding space, it can be applied to map real spinor fields to their p-adic counterparts. The natural requirement is that the modulus squared is mapped continuously in the cognitive map so that canonical identification is the natural possibility. The phases of eigenstate basis represent typically quantum numbers such as momentum components and spin. Therefore Pythagorean phases are a natural representation of the phase factors and must be mapped as such to their p-adic counterparts. Thus phase preserving canonical identification is natural for spinor fields and Schödinger amplitudes.

4   p-Adic differential and integral calculus

p-Adic differential calculus differs from its real counterpart in that piecewise constant functions depending on a finite number of pinary digits have vanishing derivative. This property implies p-adic nondeterminism, which has natural interpretation as making possible imagination if one identifies p-adic regions of space-time as cognitive regions of space-time.

One of the stumbling blocks in the attempts to construct p-adic physics have been the difficulties involved with the definition of the p-adic version of a definite integral. There are several alternative options as how to define p-adic definite integral and it is quite possible that there is simply not a single correct version since p-adic physics itself is a cognitive model.

a) The first definition of the p-adic integration is based on three ideas. The ordering for the limits of integration is defined using canonical correspondence. $x < y$ holds true if $x_R < y_R$ holds true. The integral functions can be defined for Taylor series expansion by defining indefinite integral as the inverse of the differentiation. If p-adic pseudoconstants are present in the integrand one must divide the integration range into pieces such that p-adic integration constant changes its value in the points where new piece begins.

b) Second definition is based on p-adic Fourier analysis based on the use of p-adic planewaves constructed in terms of Pythagorean phases. This definition is especially attractive in the definition of p-adic QFT limit and is discussed in detail later in the section 'p-Adic Fourier analysis'. In this case the integral is defined in the set of rationals and the ordering of the limits of integral is therefore not a problem.

c) For p-adic functions which are direct canonical images of real func-

tions, p-adic integral can be defined also as a limit of Riemann sum and this in principle makes the numerical evaluation of p-adic integrals possible. As found in the chapter 'Mathematical Ideas', Riemann sum representation leads to an educated guess for an *exact formula for the definite integral* holding true for functions which are p-adic counterparts of real-continuous functions and for p-adically analytic functions. The formula provides a calculational recipe of p-adic integrals, which converges extremely rapidly in powers of p. Ultrametricity guarantees the absence of divergences in arbitrary dimensions provided that integrand is a bounded function. It however seems that this definition of integral cannot hold true for the p-adically differentiable function whose real images are not continuous.

4.1   p-Adic differential calculus

The rules of the p-adic differential calculus are formally identical to those of the ordinary differential calculus and generalize in a trivial manner for the algebraic extensions.

The class of the functions having vanishing p-adic derivatives is larger than in the real case: any function depending on a finite number of positive pinary digits of p-adic number and of arbitrary number of negative pinary digits has a vanishing p-adic derivative. This becomes obvious, when one notices that the p-adic derivative must be calculated by comparing the values of the function at nearby points having the same p-adic norm (here is the crucial difference with respect to real case!). Hence, when the increment of the p-adic coordinate becomes sufficiently small, p-adic constant doesn't detect the variation of x since it depends on finite number of positive p-adic pinary digits only. p-Adic constants correspond to real functions, which are constant below some length scale $\Delta x = 2^{-n}$. As a consequence p-adic differential equations are non-deterministic: integration constants are arbitrary functions depending on a finite number of the positive p-adic pinary digits. This feature is central as far applications are considered and leads to the interpretation of p-adic physics as physics of cognition which involves imagination in essential manner. The classical non-determinism of the Kähler action, which is the key feature of quantum TGD, corresponds in a natural manner to the non-determinism of volition in macroscopic length scales.

p-analytic maps $g : R_p \rightarrow R_p$ satisfy the usual criterion of differentiability and are representable as power series

$$g(x) \quad = \quad \sum_k g_k x^k \quad . \tag{20}$$

Also negative powers are in principle allowed.

4.2 p-Adic fractals

p-Adically analytic functions induce maps $R_+ \to R_+$ via the canonical identification map. The simplest manner to get some grasp on their properties is to plot graphs of some simple functions (see Fig. 4.2 for the graph of p-adic x^2 and Fig. 4.2 for the graph of p-adic $1/x$). These functions have quite characteristic features resulting from the special properties of the p-adic topology. These features should be universal characteristics of cognitive representations and should allow to deduce the value of the p-adic prime p associated with a given cognitive system.

a) p-Analytic functions are continuous and differentiable from right: this peculiar asymmetry is a completely general signature of the p-adicity. As far as time dependence is considered, the interpretation of this property as a mathematical counterpart of the irreversibility looks attractive. This suggests that the transition from the reversible microscopic dynamics to irreversible macroscopic dynamics could correspond to the transition from the ordinary topology to an effective p-adic topology.

b) There are large discontinuities associated with the points $x = p^n$. This implies characteristic threshold phenomena. Consider a system whose output $f(n)$ is a function of input, which is integer n. For $n < p$ nothing peculiar happens but for $n = p$ the real counterpart of the output becomes very small for large values of p. In the bio-systems threshold phenomena are typical and p-adicity might be the key to their understanding. The discontinuities associated with the powers of $p = 2$ are indeed encountered in many physical situations. Auditory experience has the property that a given frequency ω_0 and its multiples $2^k \omega_0$, octaves, are experienced as the same frequency, this suggests that the auditory response function for a given frequency ω_0 is a 2-adicallly analytic function. Titius-Bode law states that the mutual distances of planets come in powers of 2, when suitable unit of distance is used. In turbulent systems period doubling spectrum has peaks at frequencies $\omega = 2^k \omega_0$.

c) A second signature of the p-adicity is "p-plicity" appearing in the graph of simple p-analytic functions. As an example, consider the graph of the p-adic x^2 demonstrating clearly the decomposition into p steps at each interval $[p^k, p^{k+1})$.

d) The graphs of the p-analytic functions are in general ordered fractals as the examples demonstrate. For example, power functions x^n are self-similar (the values of the function at some any interval (p^k, p^{k+1}) deter-

mines the function completely) and in general p-adic x^n with nonnegative (negative) n is smaller (larger) than real x^n expect at points $x = p^n$ as the graphs of p-adic x^2 and $1/x$ show (see Fig. 4.2 and 4.2) These properties are easily understood from the properties of the p-adic multiplication. Therefore the first guess for the behavior of a p-adically analytic function is obtained by replacing x and the coefficients g_k with their p-adic norms: at points $x = p^n$ this approximation is exact if the coefficients of the power series are powers of p. This step function approximation is rather reasonable for simple functions such as x^n as the figures demonstrate. Since p-adically analytic function can be approximated with $f(x) \sim f(x_0) + b(x - x_0)^n$ or as $a(x - x_0)^n$ (allowing non-analyticity at x_0) around any point the fractal associated with p-adically analytic function has universal geometrical form in sufficiently small length scales.

p-Adic analyticity is well defined for the algebraic extensions of R_p, too. The figures 4.2 and 4.2 visualize the behavior of the real and imaginary parts of the 2-adic z^2 function as a function of the real x and y coordinates in the parallelpiped $I^2, I = [1 + 2^{-7}, 2 - 2^{-7}]$. An interesting possibility is that the order parameters describing various phases of some physical systems are p-adically differentiable functions. The p-analyticity would therefore provide a means for coding the information about ordered fractal structures.

The order parameter could be one coordinate component of a p-adically analytic map $R^n \to R^n$, $n = 3, 4$. This is analogous to the possibility to regard the solution of the Laplace equation in two dimensions as a real or imaginary part of an analytic function. A given region V of the order parameter space corresponds to a given phase and the volume of the ordinary space occupied by this phase corresponds to the inverse image $g^{-1}(V)$ of V. Very beautiful images are obtained if the order parameter is the the real or imaginary part of a p-analytic function $f(z)$. A good example is p-adic z^2 function in the parallelpiped $[a, b] \times [a, b]$, $a = 1 + 2^{-9}$, $b = 2 - 2^9$ of C-plane. The value range of the order parameter can be divided into, say, 16 intervals of the same length so that each interval corresponds to a unique color. The resulting fractals possess features, which probably generalize to higher-dimensional extensions.

a) The inverse image is an ordered fractal and possesses lattice/cell like structure, with the sizes of cells appearing in powers of p. Cells are however not identical in analogy with the differentiation of the biological cells.

b) p-Analyticity implies the existence of a local vector valued order parameter given by the p-analytic derivative of $g(z)$: the geometric structure of the phase portrait indeed exhibits the local orientation clearly.

A second representation of the fractals is obtained by dividing the value
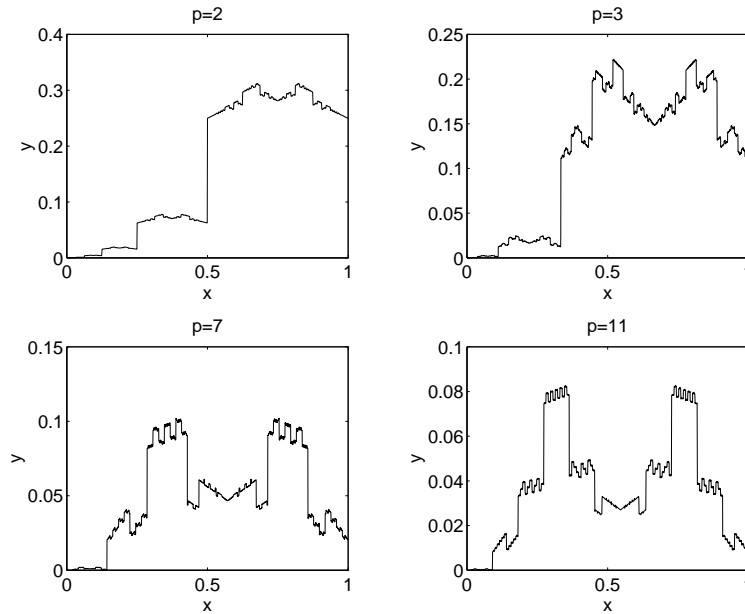
Figure 2: p-Adic x^2 function for some values of p

range of z into a finite number of intervals and associating different color to each interval. In a given resolution this representation makes obvious the presence of 0, 1- and 2-dimensional structures not obvious from the graph representation used in the figures of this book.

These observations suggests that p-analyticity might provide a means to code the information about ordered fractal structures in the spatial behavior of order parameters (such as enzyme concentrations in bio-systems). An elegant manner to achieve this is to use purely real algebraic extension for 3-space coordinates and for the order parameter: the image of the order parameter $\Phi = \phi_1 + \phi_2\theta + \phi_3\theta^2$ under the canonical identification is real and positive number automatically and might be regarded as concentration type quantity.

Figure 3: p-Adic $1/x$ function for some values of p

Figure 4: The graph of the real part of 2-adically analytic $z^2 =$ function.

z=2xy, p=2

Figure 5: The graph of 2-adically analytic $Im(z^2) = 2xy$ function.

4.3 p-Adic integral calculus

The basic problems of the integration with p-adic values of integral are caused by the facts that p-adic numbers are not well-ordered and by the properties of p-adic norm. The general idea that p-adic physics can mimic real physics only at the algebraic level, leads to the idea that p-adic integration could be a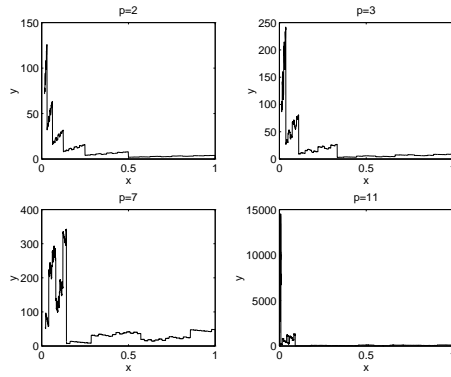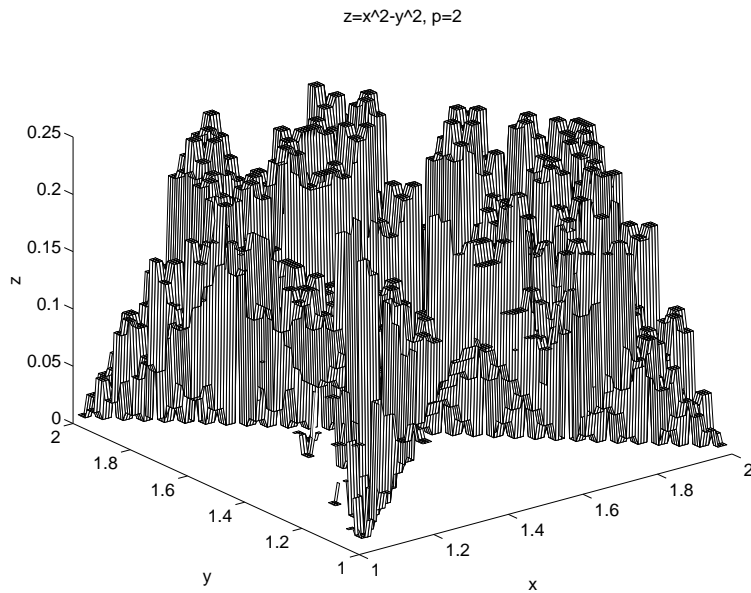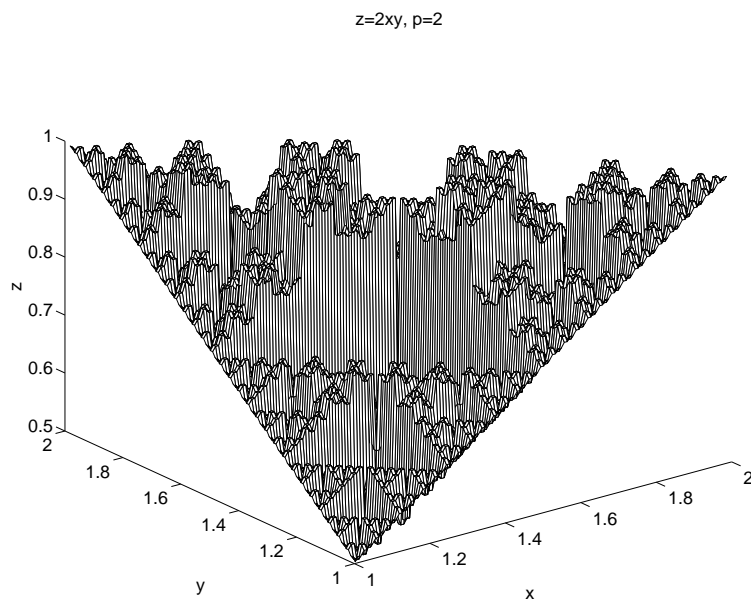lgebraized whereas numerical approaches analogous to Riemann sum are not possible. In the following three examples are discussed.

a) Definite integral can be defined using integral function and by defining integration limits via canonical identification: the drawback is the loss of general coordinate invariance. A more elegant general coordinate invariant approach is based on the identification of rationals as common to both reals and p-adics. This works for rational valued integration limits.

b) Residy calculus allows to realize integrals of analytic functions over closed curves of complex plane. The generalization of the residy calculus makes possible to realize conformal invariance at elementary particle horizons which are metrically 2-dimensional and allow conformal invariance and has also p-adic counterpart.

c) The perturbative series using Gaussian integration is the only to perform in practice infinite-dimensional functional integrals and being purely algebraic procedure, allows a straightforward p-adic generalization. This is the only option for p-adicizing configuration space integral.

4.3.1 Definition of the definite integral using integral function concept and canonical identification or identification by common rationals

The concept of the p-adic definite integral can be defined for functions $R_p \to C$ [19] using translationally invariant Haar measure for R_p. In present context one is however interested in definining a p-adic valued definite integral for functions $f : R_p \to R_p$: target and source spaces could of course be also some some algebraic extensions of the p-adic numbers.

What makes the definition nontrivial is that the ordinary definition as the limit of a Riemann sum doesn't seem to work: it seems that Riemann sum approaches to zero in the p-adic topology since, by ultra-metricity, the p-adic norm of a sum is never larger than the maximum p-adic norm for the summands. The second difficulty is related to the absence of a well-ordering for the p-adic numbers. The problems might be avoided by defining the integration essentially as the inverse of the differentiation and using the canonical correspondence to define ordering for the p-adic numbers. More

generally, the concepts of the form, cohomology and homology are crucially based on the concept of the boundary. The concept of boundary reduces to the concept of an ordered interval and canonical identification makes it indeed possible to define this concept.

The definition of the p-adic integral functions defining integration as inverse of the differentiation is straightforward and one obtains just the generalization of the standard calculus. For instance, one has $\int z^n = \frac{z^{n+1}}{(n+1)} + C$ and integral of the Taylor series is obtained by generalizing this. One must however notice that the concept of integration constant generalizes: any function $R_p \to R_p$ depending on a finite number of pinary digits only, has a vanishing derivative.

Consider next the definite integral. The absence of the well ordering implies that the concept of the integration range (a, b) is not well defined as a purely p-adic concept. As already mentioned there are two solutions of the problem.

a) The identification of rational numbers as common to both reals and p-adics allows to order the integration limits when the end points of the integral are rational numbers. This is perhaps the most elegant solution of the problem since it is consistent with the restricted general coordinate invariance allowing rational function based coordinate changes. This approach works for rational functions with rational coefficients and more general functions if algebraic extension or extension containing transcendentals like e and logarithms of primes are allowed. The extension containing e, π, and $log(p)$ is finite-dimensional if e/π and $\pi/log(p)$ are rational numbers for all primes p. Essentially algebraic continuation of real integral to p-adic context is in question.

b) An alternative resolution of the problem is based on the canonical identification. Consider p-adic numbers a and b. It is natural to define a to be smaller than b if the canonical images of a and b satisfy $a_R < b_R$. One must notice that $a_R = b_R$ does not imply $a = b$, since the inverse of the canonical identification map is two-valued for the real numbers having a finite number of pinary digits. For two p-adic numbers a, b with $a < b$, one can define the integration range (a, b) as the set of the p-adic numbers x satisfying $a \leq x \leq b$ or equivalently $a_R \leq x_R \leq b_R$. For a given value of x_R with a finite number of pinary digits, one has two values of x and x can be made unique by requiring it to have a finite number of pinary digits.

One can define definite integral $\int_a^b f(x)dx$ formally as

$$\int_a^b f(x)dx \quad = \quad F(b) - F(a) \ , \tag{21}$$

where $F(x)$ is integral function obtained by allowing only ordinary integration constants and $b_R > a_R$ holds true. One encounters however a problem, when $a_R = b_R$ and a and b are different. Problem is avoided if the integration limits are assumed to correspond to p-adic numbers with a finite number of pinary digits.

One could perhaps relate the possibility of the p-adic integration constants depending on finite number of pinary digits to the possibility to decompose integration range $[a_R, b_R]$ as $a = x_0 < x_1 < ....x_n = b$ and to select in each subrange $[x_k, x_{k+1}]$ the inverse images of $x_k \leq x \leq x_{k+1}$, with x having finite number of pinary digits in two different manners. These different choices correspond to different integration paths and the value of the integral for different paths could correspond to the different choices of the p-adic integration constant in integral function. The difference between a given integration path and 'standard' path is simply the sum of differences $F(x_k) - F(y_k)$, $(x_k)_R = (y_k)_R$.

This definition has several nice features.

a) The definition generalizes in an obvious manner to the higher dimensional case. The standard connection between integral function and definite integral holds true and in the higher-dimensional case the integral of a total divergence reduces to integral over the boundaries of the integration volume. This property guarantees that p-adic action principle leads to same field equations as its real counterpart. It this in fact this property, which drops other alternatives from the consideration.

b) The basic results of the real integral calculus generalize as such to the p-adic case. For instance, integral is a linear operation and additive as a set function.

The ugly feature is the loss of the general coordinate invariance due to the fact that canonical identification does not commute with cooordinate changes (except scalings by powers of p) and it seems that one cannot use canonical identification at the fundamental level to define definite integrals.

4.3.2 Definite integrals in p-adic complex plane using residy calculus

Residy calculus allows to calculate the integrals $\oint_C f(z)dz$ around complex curves as sums over poles of the function inside the curve:

$$\oint f(z)dz = i2\pi \sum_k Res(f(z_k)) \ , \tag{22}$$

where $Res(f(z_k))$ at pole $z = z_k$ is defined as $Res(f(z_k)) = \lim_{z \to z_k}(z - z_k)f(z)$. This definition applies in case of 2-dimensional $\sqrt{-1}$-containing algebraic extension of p-adic numbers ($p \bmod 4 = 3$) but its seems that this is not relevant for quantum TGD.

Quaternion conformal invariance corresponds to the conformal invariance associated with topologically 3-dimensional elementary particle horizons surrounding wormhole contacts which have Euclidian signature of induced metric. The induced metric is degenerate at the elementary particle horizon so that these surfaces are metrically two-dimensional. This implies a generalization of conformal invariance analogous to that at light cone cone boundary. In particular, a subfield of quaternions isomorphic with complex numbers is selected. One expects that residy calculus generalizes.

Elementary particle horizons are defined by a purely algebraic condition stating that the determinant of the induced metric vanishes, and thus the notion makes sense for p-adic space-time sheets too. Also residy calculus should make sense for all algebraic extensions of p-adic numbers and the algebra of quaternion conformal invariance would generalize to the p-adic context too. Note however that the notion of p-adic quaternions does not make sense: the reason is that p-adic Euclidian length squared for a non-vanishing p-adic quaternion can vanish so that the inverse of quaternion is not well defined always. In the set of rational numbers this failure does not however occur and this might be enough for p-adicization to work.

4.3.3 Definite integrals using Gaussian perturbation theory

In quantum field theories functional integrals are defined by Gaussian perturbation theory. For real infinite-dimensional Gaussians the procedure has a rigorous mathematical basis deriving from measure theory. For the imaginary infinite-dimensional Gaussians defining the Feynman path integrals of quantum field theory the rigorous mathematical justification is lacking.

In TGD framework the integral over the configuration space of three surface can be reduced to a real Gaussian perturbation theory around the maxima of Kähler function. The integration is over quantum fluctuating degrees of freedom defining infinite-dimensional symmetric space for given values of zero modes. According to the more detailed arguments about how to construct p-adic counterpart of real configuration space physics described in the chapter "Construction of Quantum Theory", the following conjectures are trued.

a) The symmetric space property implies that there is only one maximum of Kähler function for given values of zero modes.

b) The generalization of Duistermaat-Heecke theorem holding true in finite-dimensional case suggests that by symmetric space property the integral of the exponent of Kähler gives just the exponent of Kähler function at the maximum and Gaussian determinant and metric determinant cancel each other.

c) The fact that free Gaussian field theory corresponds to a flat symmetric space inspires the hypothesis that S-matrix elements involving configuration space spinor fields in the representations of the isometry group reduce to those given by free field theory with propagator defined by the inverse of the configuration space covariant Kähler metric evaluated in the tangent space basis defined by the isometry currents at the maximum of Kähler function. This implies that there is no perturbation series which would spoil any hopes about proving the rationality. The reduction to a free field theory does not make quantum TGD non-interacting since interactions are described as topologically (as decays and fusions of 3-surfaces) rather than algbraically as non-linearities of local action.

d) If the exponent function is a rational function with rational coefficients in the sense that for the points of configuration space having finite number of rational valued coordinates (also zero modes), then the exponent $e^{K_{max}}$ is a rational number for rational values of zero modes. From the rationality of the exponent of the Kähler function follows the rational valuedness of the matrix elements of the metric. The undeniably very optimistic conclusion is that for rational values of the zero modes the S-matrix elements would be rational valued or have values if finite extension of rationals, so that they could be continued to the p-adic sectors of the configuration space. The S-matrix would have the same form in all number fields.

e) One could also interpret the outcome as an algebraic continuation of the rational quantum physics to real and p-adic physics. Configuration space-integrals can be thought of as being performed in the rational configuration space. Of course, one can define also ordinary integrals over R^n numerically using Riemann sums by considering the division of the integration region to very small n-cubes for which the sides have rational-number valued lengths and such that the value of the function is taken at rational valued point inside each cube.

The finite-dimensional real one-dimensional Gaussian $exp(-ax^2/2)$ provides a natural testing ground for this rather speculative picture. The integral of the Gaussian is $(2\pi)^{1/2}/\sqrt{a}$: in n-dimensional case where a is replaced by a quadratic form defined by a matrix A one obtains $(2\pi)^{n/2}/\sqrt{det(A)}$ in n-dimensional case. The integral of a function $exp(-ax^2 + kx^n)x^k$ reduces to a perturbation series as sum of graphs containing single vertex contain-

ing k lines and arbitrary number of vertices containing n lines and endowed with a factor k, and assigning with the lines the propagator factor $1/a$. For n-dimensional case the propagator factor would be inverse of the matrix A.

The result makes sense in the p-adic context if a and k are rational numbers. In the n-dimensional case matrix A and the coefficients defining the polynomial defining the interaction term must be rational numbers. The only problematic factor is the power of 2π, which seems to require algebraic extension containing π. Of course, one could define the normalization of the functional integral by dividing it by $(2\pi)^{n/2}$ to get rid of this fact. In the definition of S-matrix elements this normalization factor always disappears so that this problem has no physical significance.

In the case of free scalar quantum field theory n-point functions the perturbation theory are simply products of 2-point functions defined by the inverse of the infinite-dimensional Gaussian matrix. For plane wave basis for scalar field labelled by 4-momentum k the inverse of the Gaussian matrix reduces to the propagator ($i/(k^2 + i\epsilon)$ for scalar field), which is rational function of the square of 4-momentum vector. In case of interacting quantum field the infinite summation over graphs spoils the hopes of obtaining end result which could be proven to be rational valued for rational values of incoming and outgoing four-momenta. The loop integrals are source of divergence problems and also number-theoretically problematic.

5 p-Adic symmetries and Fourier analysis

5.1 p-Adic symmetries and generalization of the notion of group

The most basic questions physicist can ask about the p-adic numbers are related to symmetries. It seems obvious that the concept of a Lie-group generalizes: nothing prevents from replacing the real or complex representation spaces associated with the definitions of the classical Lie-groups with the linear space associated with some algebraic extension of the p-adic numbers: the defining algebraic conditions, such as unitarity or orthogonality properties, make sense for the algebraically extended p-adic numbers, too.

For orthogonal groups one must replace the ordinary real inner product with the inner product $\sum_k X_k^2$ with a Cartesian power of a purely real extension of p-adic numbers. In the unitary case one must consider the complexification of a Cartesian power of a purely real extension with the inner product $\sum \bar{Z}_k Z_k$. Here $p \bmod 4 = 3$ is required. It should be emphasized however that the p-adic inner product differs from the ordinary one so that

the action of, say, p-adic counterpart of a rotation group in R_p^3 induces in R^3 an action, which need not have much to do with ordinary rotations so that the generalization is physically highly nontrivial. Extensions of p-adic numbers also mean extreme richness of structure.

The exponentiation $t \to exp(tJ)$ of the Lie-algebra element J is a central element of Lie group theory and allows to coordinatize that elements of Lie group by mapping tangent space points the points representing group elements. Without algebraic extensions involving e or its roots one can exponentiate only the group parameters t satisfying $|t|_p < 1$. Thus the values of the exponentiation parameter which are too small/large in real/p-adic sense are not possible and one can say that the standard p-adic Lie algebra is a ball with radius $|t|_p = 1/p$.

The study of ordinary one-dimensional translations gives an idea about what it is involved. For finite values of the p-adic integer t the exponentiated group element corresponds in the case of translation group to a power of e so that the points reached by exponentiation cannot correspond to rational points. Since logarithm function exist as an inverse of p-adic exponent and since rationals correspond to infinite but periodic pinary expansions, rational points having the same p-adic norm can be reached by p-adic exponentials using t which is infinite as ordinary integer. This result is expected to generalize to the case of groups represented using rational-valued matrices.

One can define a hierarchy of p-adic Lie-groups by allowing extensions allowing e and even its roots such that the algebras have p-adic radii p^k. Hence the fact that the powers $e, ..., e^{p-1}$ define a finite-dimensional extensions of p-adic numbers seems to have a deep group theoretical meaning. One can define a hierarchy of increasingly refined extensions by taking the generator of extension to be $e^{1/n}$. For instance, in the case of translation group this makes possible p-adic variant of Fourier analysis by using discrete plane wave basis.

One can generalize also the notion of group by using the generalized notion of number. This means that one starts from the restriction of the group in question to a group acting in say rational and complex rational linear space and requires that real and p-adic groups have rational group transformations as common. By performing various completions one obtains a generalized group having the characteristic book like structure. In this kind of situation the relationship between various groups is clear and also the role of extensions of p-adic numbers can be understood. The notion of Lie-algebra generalizes also to form a book like structure. Coefficients of the pages of the Lie-algebra belong to various number fields and rational valued coefficients correspond to a part partially (because of the restriction

$|t|_p < p^k$) common to all Lie-algebras.

5.1.1 $SO(2)$ as example

A simple example is provided by the generalization of the rotation group $SO(2)$. The rows of a rotation matrix are in general n orthonormalized vectors with the property that the components of these vectors have p-adic norm not larger than one. In case of $SO(2)$ this means the the matrix elements $a_{11} = a_{22} = a, a_{12} = -a_{21} = b$ satisfy the conditions

$$
\begin{aligned}
a^2 + b^2 &= 1 \ , \\
|a|_p &\leq 1 \ , \\
|b|_p &\leq 1 \ .
\end{aligned}
\tag{23}
$$

One can formally solve a as $a = \sqrt{1 - b^2}$ but the solution doesn't exists always. There are various possibilities to define the orthogonal group.

a) One possibility is to allow only those values of a for which square root exists as p-adic number. In case of orthogonal group this requires that both $b = sin(\Phi)$ and $a = cos(\Phi)$ exist as p-adic numbers. If one requires further that a and b make sense also as ordinary rational numbers, they define a Pythagorean triangle (orthogonal triangle with integer sides) and the group becomes discrete and cannot be regarded as a Lie-group. Pythagorean triangles emerge for rational counterpart of any Lie-group.

b) Other possibility is to allow an extension of the p-adic numbers allowing a square root of any ordinary p-adic number. The minimal extensions has dimension 4 (8) for $p > 2$ ($p = 2$). Therefore space-time dimension and imbedding space dimension emerge naturally as minimal dimensions for spaces, where p-adic $SO(2)$ acts 'stably'. The requirement that a and b are real is necessary unless one wants the complexification of $so(2)$ and gives constraints on the values of the group parameters and again Lie-group property is expected to be lost.

c) The Lie-group property is guaranteed if the allowed group elements are expressible as exponents of a Lie-algebra generator Q. $g(t) = exp(iQt)$. This exponents exists only provided the p-adic norm of t is smaller than one. If one uses square root allowing extension, one can require that t satisfies $|t| \leq p^{-n/2}$, $n > 0$ and one obtains a decreasing hierarchy of groups $G_1, G_2, ...$ For the physically interesting values of p (typically of order $p = 2^{127} - 1$) the real counterparts of the transformations of these groups are extremely near to the unit element of the group. These conclusions hold true for any group.

An especially interesting example physically is the group of 'small' Lorentz transformations with $t = O(\sqrt{p})$. If the rest energy of the particle is of order $O(\sqrt{p})$: $E_0 = m = m_0\sqrt{p}$ (as it turns out) then the Lorentz boost with velocity $\beta = \beta_0\sqrt{p}$ gives particle with energy $E = m/\sqrt{1 - \beta_0^2 p} = m(1 + \frac{\beta_0^2 p}{2} + ..)$ so that $O(p^{1/2})$ term in energy is Lorentz invariant. This suggests that non-relativistic regime corresponds to small Lorentz transformations whereas in genuinely relativistic regime one must include also the discrete group of 'large' Lorentz transformations with rational transformations matrices.

d) One can extend the group to contain products $G_1 G_2$, such that G_1 is a rational matrix belonging to the restriction of the Lie-group to rational matrices not obtainable from a unit matrix p-adically by exponentiation, and G_2 is a group element obtainable from unit element by exponentiation. For instance, rational CP_2 is obtained from the group of rational 3×3 unitary matrices as by dividing it by the $U(2)$ subgroup of rational unitary matrices.

Even the construction of the representations of the translation group raises nontrivial issues since the construction of p-adic Fourier analysis is by no means a nontrivial task. One can however define the concept of p-adic plane wave group theoretically and p-adic plane waves are orthogonal with respect to the inner product defined by the proposed p-adic integral.

The representations of 3-dimensional rotation group $SO(3)$ can be constructed as homogenous functions of Cartesian coordinates of E^3 and in this case the phase factors $exp(im\phi)$ typically appearing in the expressions of spherical harmonics do not pose any problems. The construction of p-adic spherical harmonics is possible if one assumes that allowed spherical angles (θ, ϕ) correspond to Pythagorean triangles.

A similar situation is encountered also in the case of CP_2 spherical harmonics in in fact, quite generally. This number theoretic quantization of angles could be perhaps interpreted as a kind of cognitive quantum effect consistent with the fact that only rationals can be visualized concretely and relate directly to the sensory experience. More generally, the possibility to realize only rationals numerically might reflect the facts that only rationals are common to reals and p-adics and that cognition is basically p-adic.

5.1.2 Fractal structure of the p-adic Poincare group

p-Adic Poincare group, just as any other p-adic Lie group, contains entire fractal hierarchy of subgroups with the same Lie-algebra. For instance, translations $m^k \rightarrow m^k + p^N a^k$, where a^k has p-adic norm not larger than

one form subgroup for all values of N. The larger the value of N is, the smaller this subgroup is. Quite generally this implies orbits within orbits and representations within representations like structure so that p-adic symmetry concept contains hologram like aspect. This property of the p-adic symmetries conforms nicely with the interpretation of p-adic symmetries as cognitive representations of real symmetries since the symmetries can be realized in a p-adically finite spatiotemporal volume of the cognitive space-time sheet. Even more, this volume can be p-adically arbitrarily small. If one identifies both p-adics and reals as a completion of rationals, the corresponding real volumes are however strictly speaking infinite in absence of a pinary cutoff.

The hierarchy of subgroups implies that M_+^4 decomposes in a natural manner to 4-cubes with side $L_0 = N_p(L)L_p$, where $N_p(L) = p^{-N}$ denotes the p-adic norm of L such that these 4-cubes are invariant under the group of sufficiently small Poincare transformations. In real context these cubes define a hierarchy of exteriors of cubes with decreasing sizes. One can have full p-adic Poincare invariance in p-adically arbitrarily small volume. Only those Poincare transformations, which leave the minimal p-adic cube invariant are symmetries. Also this picture suggest that the p-adic space-time sheets providing cognitive representations about finite space-time regions by canonical identification can have very large size.

The construction of the p-adic Fourier analysis is a nontrivial problem. The usual exponent functions $f_P(x) = exp(iPx)$, providing a representation of the p-adic translations do not make sense as a Fourier basis: f_P is not a periodic function; f_P does not converge if the norm of Px is not smaller than one and the natural orthogonalization of the different momentum eigenstates does not seem to be possible using the proposed definition of the definite integral.

This state of affairs suggests that p-adic Fourier analysis involves number theory. It turns out that one can construct what might be called number theoretical plane waves and that p-adic momentum space has a natural fractal structure in this case. The basic idea is to reduce p-adic Fourier analysis to a Fourier analysis in a finite field $G(p,1)$ plus fractality in the sense that all p^m-scaled versions of the $G(p,1)$ plane waves are used. This means that p-adic plane waves in a given interval $[n, n+1)p^m$ are piecewise constant plane waves in a finite field $G(p,1)$. Number theoretical p-adic plane waves are pseudo constants so that the construction does not work for p-adically differentiable functions. The pseudo-constancy however turns out to be a highly desirable feature in the construction of the p-adic QFT limit of TGD based on the mapping of the real H-quantum fields to p-adic

quantum fields using the canonical identification.

The unsatisfactory feature of this approach is that number theoretic p-adic plane waves do not behave in the desired manner under translations. It would be nice to have a p-adic generalization of the plane wave concept allowing a generalization of the standard Fourier analysis and a direct connection with the theory of the representations of the translation group. A natural idea is to to define exponential function as a solution of a p-adic differential equation representing the action of a translation generator and to introduce multiplicative pseudo constant making possible to define exponential function for all values of its argument. One can develop an argument suggesting that the plane waves obtained in this manner are indeed orthogonal.

Infinitesimal form of translational symmetry might be argued to be too strong requirement since p-adically infinitesimal translations typically correspond to real translations which are arbitrarily large: this is not consistent with the idea that cognitive representations with a finite spatial resolution are in question. This motivates a third approach to the p-adic Fourier analysis. The basic requirement is that discrete subgroup of translations commutes with the map of the real plane waves to their p-adic counterparts. This means that the products of the real phase factors are mapped to the products of the corresponding p-adic phase factors. This is possible if the phase factor is a rational complex number so that the phase angle corresponds to a Pythagorean triangle. The p-adic images of the real plane waves are defined for the momenta $k = nk_G$, $k_G = \phi_G/\Delta x$, where $\phi_G \in [0, 2\pi]$ is a Pythagorean phase angle and where the points $x_n = n\Delta x$ define a discretization of x-space, Δx being a rational number. These plane waves form a complete and orthogonalized set.

5.2   p-Adic Fourier analysis: number theoretical approach

Contrary to the original expectations, number theoretical Fourier analysis is probably not basic mathematical tools of p-adic QFT since it fails to provide irreducible representation for the translational symmetries. Despite this it deserves documentation.

5.2.1   Fourier analysis in a finite field $G(p, 1)$

The p-adic numbers of unit norm modulo p reduce to a finite field $G(p, 1)$ consisting of the integers $0, 1, ..., p-1$ with arithmetic operations defined by those of the ordinary integers taken modulo p. Since the elements $1, ..., p-$

1 form a multiplicative group there must exists an element a of $G(p,1)$ (actually several) such that $a^{p-1} = 1$ holds true in $G(p,1)$. This kind of element is called primitive root. If n is a factor of $p-1$: $(p-1) = nm$, then also $a^m = 1$ holds true. This reflects the fact that Z_{p-1} decomposes into a product $Z_{m_1}^{n_1} Z_{m_2}^{n_2} ... Z_{m_s}^{n_s}$ of commuting factors Z_{m_i}, such that $m_i^{n_i}$ divides $p-1$.

A Fourier basis in $G(p,1)$ can be defined using p functions $f_k(n)$, $k = 0, .., p-1$. For $k = 0, 1, ..., p-2$ these functions are defined as

$$f_k(n) \quad = \quad a^{nk} \quad , \quad n = 0 \ , ..., p-1 \ , \tag{24}$$

and satisfy the periodicity property

$$f_k(0) = f_k(p-1) \ .$$

The problem is to identify the lacking p:th function. Since $f_k(n)$ transforms irreducibly under translations $n \to n + m$ it is natural to require that also the p:th function transforms in a similar manner and satisfies the periodicity property. This is achieved by defining

$$f_{p-1}(n) \quad = \quad (-1)^n \ . \tag{25}$$

The counterpart of the complex conjugation for f_k for $k \neq p-1$ is defined as $f_k \to f_{p-1-k}$. f_{p-1} is invariant under the conjugation. The inner product is defined as

$$\langle f_k, f_l \rangle = \sum_{n=0}^{p-2} f_{p-1-k}(n) f_l(n) = \delta(k,l)(p-1) \ . \tag{26}$$

The dual basis $\hat{f}_k$ clearly differs only by the normalization factor $1/(p-1)$ from the basis f_{p-k}. The counterpart of Fourier expansion for any real function in $G(p,1)$ can be obviously constructed using this function basis and Fourier components are obtained as the inner products of the dual Fourier basis with the function in question.

A natural interpretation for the integer k is as a p-adic momentum since in the translations $n \to n + m$ the plane wave with $k \neq p-1$ changes by a phase factor a^{km}. For $k = p-1$ it transforms by $(-1)^m$ so that also now an eigen state of finite field translations is in question.

5.2.2　p-Adic Fourier analysis based on p-adic plane waves

The basic idea is to reduce p-adic Fourier analysis to the Fourier analysis in $G(p, 1)$ by using fractality.

a) Let the function $f(x)$ be such that the maximum p-adic norm of $f(x)$ is p^{-m}. One can uniquely decompose $f(x)$ to a sum of functions $f_n(x)$ such that $|f_n(x)|_p = p^n$ or vanishes in the entire range of definition for f:

$$
\begin{aligned}
f(x) &= \sum_{n \geq m} f_n(x) \ , \\
f_n(x) &= g_n(x)p^n \ , \\
|g_n(x)| &= 1 \ \text{for} \ g(x) \neq 0 \ .
\end{aligned} \tag{27}
$$

The higher the value of n, the smaller the contribution of f_n. The expansion converges extremely rapidly for the physically interesting large values of p.

b) Assume that $f(x)$ is such that for each value of n one can find some resolution $p^{m(n)}$ below which $g_n(x)$ is constant in the sense that for all intervals $[r, r+1)p^{m(n)}$ (defined in terms of the canonical identification) the function $f_n(x)$ is constant. For p-adically differentiable functions this cannot be the case since they would be pseudo constants if this were true. In the physical situation CP_2 size provides a natural p-adic cutoff so that only a finite number of f_n:s are needed and the resolution in question corresponds to CP_2 length scale. Hence ordinary plane waves (possibly with a natural UV cutoff) should have an expansion in terms of the p-adic plane waves.

c) The assumption implies that in each interval $(r, r+1)p^{m(n)-1}$, g_n can be regarded as a function in $G(p, 1)$ identified as the set $x = (r+sp)p^{m(n)-1}$, $s = 0, 1, ..., p-1$. Hence one can Fourier expand $f_n(x)$ using $G(p, 1)$ plane waves f^{ks}. In this manner one obtains a rapidly converging expansion using p-adic plane waves.

5.2.3　Periodicity properties of the number theoretic p-adic plane waves

The periodicity properties of the p-adic plane waves make it possible to associate a definite wavelength with a given p-adic plane wave. For the p-adic momenta k not dividing $p-1$, the wavelength corresponds to the entire range $(n, n+1)p^m$ and its real counterpart is

$$
\lambda = p^{-m-1/2}l \ ,
$$

where $l \sim 10^4 \sqrt{G}$ is the fundamental p-adic length scale. If k divides $p-1 = \prod_i m_i^{n_i}$, the period is m_i and the real wavelength is

$$\lambda(m_i) = m_i p^{-m-1-1/2} l \ .$$

One might wonder whether this selection of preferred wavelengths has some physical consequences. The first thing to notice is that p-adic plane waves do *not* replace ordinary plane waves in the construction of the p-adic QFT limit of TGD. Rather, ordinary plane waves are expanded using the p-adic plane waves so that the selection of the preferred wavelengths, if it occurs at all, must be a dynamical process. The average value of the prime divisors, and hence the number of different wavelengths for a given value of p, counted with the degeneracy of the divisor is given by [22]

$$\Omega(n) = ln(ln(n)) + 1.0346 \ ,$$

and is surprisingly small, or order 6 for numbers of order M_{127}! If one can apply probabilistic arguments or [22] to the numbers of form $p-1$, too then one must conclude that very few wavelengths are possible for general prime p! This in turn means that to each p there are associated only very few characteristic length scales, which are predictable. Furthermore, all the p^k-multiples of these scales are also possible if p-adic fractality holds true in macroscopic length scales.

Mersenne primes M_n can be considered as an illustrative example of the phenomenon. From [23] one finds that $M_{127} - 1$ has 11 distinct prime factors and 3 and 7 occurs three and 2 times respectively. The number of distinct length scales is $3 \cdot 2^{11} - 1 \sim 2^{12}$. $M_{107} - 1$ and $M_{89} - 1$ have 7 and 11 singly occurring factors so that the numbers of length scales are $2^7 - 1 = 127 = M_7$ and $2^{11} - 1$. Note that for hadrons (M_{107}) the number of possible wavelengths is especially small: does this have something to do with the collective behavior of color confined quarks and gluons? An interesting possibility is that this length scale generation mechanism works even macroscopically (for p-adic length scale hypothesis at macroscopic length scales see the third part of the book). One cannot exclude the possibility that long wavelength photons, gravitons and neutrinos might therefore provide a completely new mechanism for generating periodic structures with preferred sizes of period.

5.3 p-Adic Fourier analysis: group theoretical approach

The problem with the straightforward generalization of the Fourier analysis is that the standard Taylor expansion of the plane wave $exp(ikx)$ converges only provided x has p-adic norm smaller than one and that the p-adic exponential function does not have the periodicity properties of the ordinary exponential function guaranteing orthogonality of the functions of the Fourier basis. Besides this one must assume $p \ mod \ 4 = 3$ to guarantee that $\sqrt{-1}$ does not exist as ordinary p-adic number.

5.3.1 The approach based on algebraic extensions allowing trigonometry

In an attempt to construct Fourier analysis the safest approach is to start from the ordinary Fourier analysis at circle or that for a particle in a one-dimensional box. The function basis uses as the basic building blocks the functions $e^{in\phi}$ in the case of circle and functions $e^{in\pi x/L}$ in the case of a particle in a box of side L.

The view about rationals as common to both reals and p-adics, and the possibility of finite-dimensional extensions of p-adics generated by the roots $e^{i2\pi/p^k}$ suggest how to realize this idea.

a) Consider first the case of the circle. Fix some value of N and select a set of points $\phi_n = in2\pi/p^k$ at which the phases are defined meaning p^{k+1}-dimensional algebraic extension. That powers of p appear is consistent with p-adic fractality. If so spin $1/2$ *resp.* spin 1 particles would be inherently 2-adic *resp.* 3-adic. The plane wave basis corresponds $exp(ik\phi_n)$, $k = 0, ..., N-1$. In the case of particle in the one-dimensional box such that L corresponds to a rational number, the box is decomposed into N intervals of length L/N.

b) One can assign to the phases a well defined angular momentum as integer $n = 0, ..., N-1$ whereas the momentum spectrum for a particle in a box are given by $n\pi/L$. It is possible to continue the phase factor to the neighborhood of each point by requiring that the differential equation

$$\frac{d}{dx}exp(ikx) = ikexp(ikx)$$

defining the exponential function is satisfied.

c) The inner product of the plane waves $f_{k_1})$ and f_{k_2} can be defined as the sum

$$\langle k_1 \rangle \equiv \sum_n \overline{f}_{k_1}(x_n) f_{k_2}(x_n) \ , \qquad (28)$$

and orthogonality and completeness differ by no means from those of ordinary Fourier analysis.

5.3.2 p-Adic Fourier analysis, Pythagorean phases, and Gaussian primes

An alternative approach is based on Pythagorean phases and discretization in x-space, which is very natural thing to do if p-adic field theory is taken as a cognitive model rather than 'real' physics. This is also natural because rational Minkowski space is in the algebraic approach the fundamental object and reals and p-adics emerge as its completions.

Rational phase factors are common to the complexified p-adics ($p \bmod 4 = 3$) and reals and this suggests that one should define p-adic plane waves so that their values are in the set of the Pythagorean phases. Pythagorean phases are in one-one correspondence with the phases of the squares of Gaussian integers N_G and thus generated as products of squares of Gaussian primes π_G, which are complex integers with modulus squared equal to prime $p \bmod 4 = 1$. Thus the set of phases $\phi(\pi_G)$ for the phases for π_G^2 form an algebraically infinite-dimensional linear space in the sense that the phases representable as superpositions

$$2\phi_G = \sum_{\pi_G} n_{\pi_G} 2\phi(\pi_G)$$

of these phases with integer coefficients belong to the set.

Consider now the definition of the plane wave basis based on Pythagorean phases and the identification of the p-adics and reals via common rationals.

a) Let $x_0 = q = m/n$ denote a value of x-coordinate and let k denote some value of momentum. If $exp(ikx_0)$ is a Pythagorean phase then also the multiples nk correspond to Pythagorean phases. k itself cannot be a rational number so that k is not defined as an ordinary p-adic number: this could be seen as a defect of the approach since one cannot speak of a well-defined momentum. Neither can k be a rational multiple of π so that Pythagorean phases have nothing to do with the phases defined by algebraic extensions containing the phase $exp(i\pi/n)$ already discussed.

For a given value of $x_0 = q$ the momenta k for which $exp(ikq)$ is a Pythagorean phase are in one-one correspondence with Pythagorean phases.

Moreover, Pythagorean phases result in the lattice defined by the multiples of the x_0. Thus a natural definition of the p-adic plane waves emerges predicting a maximal momentum spectrum with one-one correspondence with Pythagorean phases, and selecting a preferred lattice of points at the real axis. This definition is also in accordance with the idea that p-adic plane waves are related with a cognitive representation for real physics.

b) Pythagorean phases are in one-one correspondence with the phase factors associated with the squares of the Gaussian integers and generating phases correspond to the phases $\phi(\pi_G)$ associated with the squares of Gaussian primes π_G. The moduli squared for the Gaussian primes correspond to squares of rational primes $p \; mod \; 4 = 1$. Thus set of allowed momenta k_G for given spatial resolution m/n is the set

$$\{k_G(q)\} = \{\tfrac{2\phi_G/}{q} + \tfrac{2\pi n}{q} | n \in Z\} \ ,$$

$$\{\phi_G\} = \{\textstyle\sum_{\pi_G} n_{\pi_G} \phi(\pi_G)\} \ .$$

When the spatial resolution $x_0 = q$ is replaced with $q_1 = r/s$, the spectrum is scaled by a rational factor q/q_1. The set of momenta is a dense subset of the real axis. There is no observable difference between the real momenta differing by a multiple of $2\pi/q$ and one must drop them from consideration. This conclusion is forced also by the fact that p-adically the momenta $k = nk_0$ do not exist, it is only the phase factors which exist.

c) It is easy to see that the p-adic plane waves with different momenta are orthogonal to each other as complex rational numbers:

$$\sum_n exp \left[in(k_G(1) - k_G(2)) \right] = 0 \ .$$

d) Also completeness relations are satisfied in the sense that the condition

$$\sum_{k_G} exp \left[i(n_1 - n_2)k_G \right] = 0$$

is satisfied for $n_1 \neq n_2$. This is due to the fact that all integer multiples of k_G define Pythagorean phases. This means that the Fourier series of a function with respect to Pythagorean phases makes sense and one can expand p-adic-valued functions of space-time coordinates as Fourier series using Pythagorean phases. In particle expansion of the the imbedding space coordinates as functions of p-adic space-time coordinates might be carried out in this manner.

e) One can criticise this approach for the fact that there is no unique continuation of the phase factors from the set of the rationals $x_n = nx_0$ to p-adic numbers neighborhoods of these points. Although eigen states of finite translations are in question one cannot regard the states as eigen states of infinitesimal translations since the momenta are not well defined as p-adic numbers. One could of course arbitrarily assign momentum eigenstate $e^{in\pi(x-x_k)}$ the point x_k to the eigenstate characterized by the dimensionless momentum n but the momentum spectrum associated with different Pythagorean phases would be same.

6 Generalization of Riemann geometry

In real context the coordinatization of manifold is regarded as a trivial problem. It took long time to realize that in p-adic context the proper treatment of coordinatization problem leads to deep insights about p-adic symmetries and about the origin og the p-adic length scales hypothesis. There are several approaches to the construction of the p-adic Riemann geometry. The most simple minded approach relies on a direct generalization of the real line element and to the proposed integral for p-adically analytic functions. A more refined approach relies on the general physical consistency conditions provided by quantum TGD and by the proposed definition of the Riemann integral.

6.1 p-Adic Riemannian geometry as a direct formal generalization of real Riemannian geometry

It is possible to generalize the concept of the (sub)manifold geometry to a p-adic (sub)manifold geometry and it seems that this definition of p-adic geometry indeed works at the level of the imbedding space. The formal definition of p-adic Riemannian geometry is based on p-adic line element

$$ds^2 = g_{kl}dx^k dx^l \ .$$

The minimal requirement is that inner products of tangent space vectors exist. Lengths and angles are defined in the usual manner.

A stronger and somewhat questionable requirement is that also curve lengths, areas, volumes, etc.. exist. This requires the definition of the square root ds of the line element. In general case the existence of a square root forces an extension of the p-adic numbers allowing square roots of ordinary p-adic numbers. As found, the extension is 4-dimensional for $p > 2$

and 8-dimensional in $p = 2$ case. It must be emphasized that the algebraic dimensions do not have interpretation as physical dimensions. The extension in question must appear as a coefficient ring of the p-adic tangent space so that p-adic Riemann spaces must be locally Cartesian powers of $4-$ $(p > 2)$ or 8-dimensional $(p = 2)$ extension. Therefore the TGD:eish dimensions of the space-time and imbedding space emerge very naturally in the p-adic context. In order to avoid the appearance of an imaginary unit in $p \bmod 4 = 3$ case, one must multiply ds^2 with -1 if the square root of $(\frac{ds}{dt})^2$ is imaginary so that one has

$$s = \int ds = \int \sqrt{\epsilon g_{kl} \frac{dx^k}{dt} \frac{dx^l}{dt}} dt \ ,$$

where ϵ is a sign factor. The p-adic length of a curve can be calculated if the integrand is integrable in the sense defined previously.

The definition of a pseudo-Riemannian metric poses problem: it seems that one should be able to make distinction between negative and positive p-adic numbers. A possible manner to make this distinction is to define p-adic numbers with unit norm to be positive or negative according to whether they are squares or not. This definition makes sense if -1 does not possess square root: this is true for $p \bmod 4 = 3$. This condition will be encountered in most applications of the p-adic numbers. At analytic level the definition generalizes in an obvious manner: what is required that the components of the metric are ordinary p-adic numbers. The p-adic counter part of the Minkowski metric can be defined as

$$ds_p^2 \;\; = \;\; (dm^0)^2 - ((dm^1)^2 + (dm^2)^2 + (dm^3)^2) \ . \tag{29}$$

The real image of this line element under canonical identification is non-negative but the metric allows to define the p-adic counterpart of M^4 light-cone as the surface $(m^0)^2 - ((m^1)^2 + (m^2)^2 + (m^3)^2) = 0$ and this surface can be regarded as a fractal counterpart of the ordinary light cone. Furthermore, this metric allows the p-adic counterpart of the Lorentz group as its group of symmetries.

The p-adic length of a curve can be finite also in the case when the real length diverges. This is the case for fractal curves contained in a finite volume of space: coast of Britain is the canonical example. The reason is that by p-adic ultra-metricity p-adic length is necessarily bounded. It is not clear whether the generalized p-adic Riemann sum has well defined limit for curves, which are general fractals. An interesting possibility is that one

could define the length of a fractal curve ('coast line of Britain') using p-adic Riemannian geometry. A possible model of this curve is obtained by identifying the ordinary real plane with its p-adic counterpart via the canonical identification and modelling the fractal curve with p-adically analytic curve $x = x(t)$. The real counterpart of this curve is certainly a fractal and need not have a well defined real length. The p-adic length of this curve can be defined as the p-adic integral of $s_p = \int ds$ and its real counterpart s_R obtained by the canonical identification can be defined to be the real length of the curve.

p-Adic Riemann geometry has some special features resulting from ultrametricity. For instance, the real counterpart for the p-adic length can be longer for a portion of a curve than for the entire curve! A good example is the p-adic length for the portion $(0 < x < 1, y > 0)$ of the unit circle $x^2 + y^2 = 1$, which can be written as

$$s(\phi) = arcsin(x) \ .$$

$arcsin(x = 1)$ is not well defined p-adically so that one must actually define the p-adic counterpart of $x_R = 1$ as $x = -p$. The length of a quadrant is $s(\pi/2) = arcsin(-p)$ so that the length of a half circle is $s(\pi) = 2arcsin(-p)$. In order $O(p)$ the length of a quadrant is $s(\pi/2) =\simeq -p \simeq (p-1)p$ whereas the length of a semicircle is $s(\pi) \simeq -2p \simeq (p-2)p$ so that the real counterpart $s_R(\pi) \simeq (p-2)/p$ for the p-adic length of a half circle is shorter than the length $s_R(\pi/2) \simeq (p-1)/p$ of a quadrant for sufficiently large values of p! For very large values of p the lengths are identical in excellent approximation. If one uses the length of a quadrant as a definition of p-adic $\pi/2$ one has "$\pi/2$" $= -arcsin(p)$ which gives for the real counterpart of p-adic "$\pi/2$"$_R$: $("\pi/2")_R \simeq 1$ for large values of p.

6.2 Topological condensate as a generalized manifold

It seems that the concept of the p-adic Riemann manifold is not as such enough for the mathematization of the topological condensate concept. This manifold can be given locally p-adic topology but decomposes into regions with different values of the p-adic prime p. Also real regions are possible. These regions are glued together along their boundaries.

One can consider two possibilities for performing the identification map. Gluing together along common rationals at the boundaries defined by the rational topology is the first option, and certainly the fundamental one if one assumes that space-times are surfaces in a rational imbedding space which can be completed to either real or p-adic imbedding space. This kind

of gluing operation is very natural for the solutions of the field equations obtained by a completion of rationals to various number fields in which the power series representing the solution of the field equations converge. This will be discussed in detail in the chapter "TGD as a generalized number theory".

The second option is the use of canonical identification map or some generalization of this map mapping real space-time regions to their p-adic counterparts. This gluing operation makes sense in case of cognitive representations and is not so fundamental. In this case p-adic space-time surfaces, possibly characterized by different value of prime p, are like different sheets of a chart having common overlap region. Although the p-adic regions can be disjoint they correspond to cognitive images of the real regions such that some overlap region is mapped to the both p-adic chart sheets. This common region defines the gluing of the p-adic surfaces together.

If one requires that the p-adic space-time surface is differentiable and even more, satisfies the p-adic counterparts of the field equations, one must loosen the cognitive mapping so that the image of the real space-time surface is discrete. Therefore one must weaken also the gluing conditions by introducing pinary cutoff.

6.3 p-Adic conformal geometry?

It would be nice to have a generalization of the ordinary conformal geometry to the p-adic context. A possibility worth of studying is that the induced Kähler form defining a Maxwell field on the space-time surface, could be the basic entity of the 4-dimensional conformal geometry rather than metric. If the existence of square root is required the dimension of this geometry is $D = 4$ of $D = 8$ depending on the value of p. In the following it is assumed that the extension used is the minimal extension allowing square root and $p \ mod \ 4 = 3$ condition holds so that the imaginary unit belongs to the generators of the extension.

In 2-dimensional case line element transforms by a conformal scale factor in p-analytic map $Z \to f(Z)$. In the four-dimensional case this requirement leads to a degenerate line element

$$
\begin{aligned}
ds^2 &= g(Z, Z_c, ...)dZdZ_c \ , \\
&= g(Z, Z_c, ..)(dx^2 + dy^2 + p(du^2 + dv^2) + 2\sqrt{p}(dxdu + dydv)) \ (30)
\end{aligned}
$$

where the conformal factor $g(Z, Z_c, ..)$ is invariant under the complex conjugation. The metric tensor associated with the line element does not possess

an inverse. This is obvious from the fact that the line element depends on two coordinates Z, Z_c only so that the p-adic conformal metric is effectively 2-dimensional rather than 4-dimensional. It therefore seems that one must give up conformal covariance requirement for the line element.

In two-dimensional conformal geometry angles are the simplest conformal invariants and are expressible in terms of the inner product. In 4-dimensional case one can define invariants, which are analogous to angles. Let A and B be two vectors in the 4-dimensional quadratic extension allowing a square root. Denote A (B) and its various conjugates by A_i (B_i), $i = 1, 2, 3, 4$. Define phase like quantities $X_{ij} = $ "$exp(i2\Phi_{ij})$" between A and B by the following formulas

$$X_{ij} \equiv \frac{A_i A_j B_k B_l}{\sqrt{A_1 A_2 A_3 A_4}\sqrt{B_1 B_2 B_3 B_4}} \; . \tag{31}$$

where i, j, k, l is permutation of $1, 2, 3, 4$. Each quantity X_{ij} is invariant under one of the conjugations $_c$, $\hat{}$ or $\hat{}_c$ and X_{ij} has values in 2-dimensional subspace of the 4-dimensional extension. As in ordinary case the angles are invariant under conjugation and this means that only 3 angle like quantities exists: this is in accordance with the fact that 3-angles are needed to specify the orientation of the vector A with respect to the vector B.

One can define also more general invariants using four vectors A, B, C, D and permutations i, j, k, l and r, s, t, u of $1, 2, 3, 4$

$$\begin{aligned} U_{ijkl} &= \frac{X_{ijkl}}{X_{rstu}} \; , \\ X_{ijkl} &\equiv A_i B_j C_k D_l \; . \end{aligned} \tag{32}$$

The number of the functionally independent invariants is reduced if various conjugates of the invariants are not counted as different invariants. If 2 or 3 vectors are identical one obtains as a special case invariants associated with 3 and 2 vectors. If there are only two vectors the number of the functionally independent invariants is 6.

There exists quadratic conformal covariants associated with tensors of weight two. The general form of the covariant is given by

$$X = g^{ij:kl} A_{ij} B_{kl} \; . \tag{33}$$

The tensor $g^{ij:kl}$ has the property that in complex coordinates $Z, \bar{Z}, \hat{Z}, \hat{\bar{Z}}$ the only nonvanishing components of the tensor have $i \neq j \neq k \neq l$. This guarantees the multiplicative transformation property in the conformal transformations $Z \rightarrow W(Z)$:

$$X(W) \quad = \quad \frac{dW}{dZ}\frac{d\bar{W}}{d\bar{Z}}\frac{d\hat{W}}{d\hat{Z}}\frac{d\hat{\bar{W}}}{d\hat{\bar{Z}}}X(Z) \ . \tag{34}$$

The simplest example of tensor $g^{ij:kl}$ is permutation symbol and the instanton density of any gauge field defines a p-adic conformal covariant (the quantity is actually $Diff^4$ invariant).

7 Appendix: p-Adic square root function and square root allowing extension of p-adic numbers

The following arguments demonstrate that the extension allowing square roots of ordinary p-adic numbers is 4-dimensional for $p < 2$ and 8-dimensional for $p = 2$.

7.1 $p > 2$ resp. $p = 2$ corresponds to $D = 4$ resp. $D = 8$ dimensional extension

What is important is that only the square root of ordinary p-adic numbers is needed: the square root need not exist outside the real axis. It is indeed impossible to find a finite-dimensional extension allowing square root for all ordinary p-adic numbers numbers. For $p > 2$ the minimal dimension for algebraic extension allowing square roots near real axis is $D = 4$. For $p = 2$ the dimension of the extension is $D = 8$.

For $p > 2$ the form of the extension can be derived by the following arguments.

a) For $p > 2$ a p-adic number y in the range $(0, p - 1)$ allows square root only provided there exists a p-adic number $x \in \{0, p - 1\}$ satisfying the condition $y = x^2 \ mod \ p$. Let x_0 be the smallest integer, which does not possess a p-adic square root and add the square root θ of x_0 to the number field. The numbers in the extension are of the form $x + \theta y$. The extension allows square root for every $x \in \{0, p - 1\}$ as is easy to see. p-adic numbers $mod \ p$ form a finite field $G(p, 1)$ [21] so that any p-adic number y, which does not possess square root can be written in the form $y = x_0 u$, where u possesses square root. Since θ is by definition the square root of x_0 then

also y possesses square root. The extension does not depend on the choice of x_0.

The square root of -1 does not exist for $p \bmod 4 = 3$ [24] and $p = 2$ but the addition of θ gurantees its existence automatically. The existence of $\sqrt{-1}$ follows from the existence of $\sqrt{p-1}$ implied by the extension by θ. $\sqrt{(-1+p)-p}$ can be developed in power in powers of p and series converges since the p-adic norm of coefficients in Taylor series is not larger than 1. If $p-1$ does not possess a square root, one can take θ to be equal to $\sqrt{-1}$.

b) The next step is to add the square root of p so that the extension becomes 4-dimensional and an arbitrary number in the extension can be written as

$$Z = (x + \theta y) + \sqrt{p}(u + \theta v) \ . \tag{35}$$

In $p = 2$ case 8-dimensional extension is needed to define square roots. The addition of $\sqrt{2}$ implies that one can restrict the consideration to the square roots of odd 2-adic numbers. One must be careful in defining square roots by the Taylor expansion of square root $\sqrt{x_0 + x_1}$ since n:th Taylor coefficient is proportional to 2^{-n} and possesses 2-adic norm 2^n. If x_0 possesses norm 1 then x_1 must possess norm smaller than $1/8$ for the series to converge. By adding square roots $\theta_1 = \sqrt{-1}, \theta_2 = \sqrt{2}$ and $\theta_3 = \sqrt{3}$ and their products one obtains 8-dimensional extension.

The emergence of the dimensions $D = 4$ and $D = 8$ for the algebraic extensions allowing the square root of an ordinary p-adic number stimulates an obvious question: could one regard space-time as this kind of an algebraic extension for $p > 2$ and the imbedding space $H = M_+^4 \times CP_2$ as a similar 8-dimensional extension of the 2-adic numbers? Contrary to the first expectations, it seems that algebraic dimension cannot be regarded as a physical dimension, and that quaternions and octonions provide the correct framework for understanding space-time and imbedding space dimensions. One could perhaps say that algebraic dimensions are additional dimensions of the world of cognitive physics rather than those of the real physics and there presence could perhaps explain why we can imagine all possible dimensions mathematically.

By construction, any ordinary p-adic number in the extension allows square root. The square root for an arbitrary number sufficiently near to p-adic axis can be defined through Taylor series expansion of the square root function $\sqrt{Z}$ at a point of p-adic axis. The subsequent considerations show that the p-adic square root function does not allow analytic continuation to

R^4 and the points of the extension allowing a square root consist of disjoint converge cubes forming a structure resembling future light cone in certain respects.

7.2 p-Adic square root function for $p > 2$

The study of the properties of the series representation of a square root function shows that the definition of the square root function is possible in certain region around the real p-adic axis. What is nice that this region can be regarded as the p-adic analog (not the only one) of the future light cone defined by the condition

$$N_p(Im(Z)) \quad < \quad N_p(t = Re(Z)) = p^k \ , \tag{36}$$

where the real p-adic coordinate $t = Re(Z)$ is identified as a time coordinate and the imaginary part of the p-adic coordinate is identified as a spatial coordinate. The p-adic norm for the four-dimensional extension is analogous to ordinary Euclidian distance. p-Adic light cone consists of cylinders parallel to time axis having radius $N_p(t) = p^k$ and length $p^{k-1}(p-1)$. As a real space (recall the canonical correspondence) the cross section of the cylinder corresponds to a parallelpiped rather than ball.

The result can be understood heuristically as follows.

a) For the four-dimensional extension allowing square root $(p > 2)$ one can construct square root at each point $x(k, s) = sp^k$ represented by ordinary p-adic number, $s = 1, ..., p - 1$, $k \in Z$. The task is to show that by using Taylor expansion one can define square root also in some neighbourhood of each of these points and find the form of this neighbourhood.

b) Using the general series expansion of the square root function one finds that the convergence region is p-adic ball defined by the condition

$$N_p(Z - sp^k) \quad \leq \quad R(k) \ , \tag{37}$$

and having radius $R(k) = p^d, d \in Z$ around the expansion point.

c) A purely p-adic feature is that the convergence spheres associated with two points are either disjoint or identical! In particular, the convergence sphere $B(y)$ associated with any point inside convergence sphere $B(x)$ is identical with $B(x)$: $B(y) = B(x)$. The result follows directly from the ultra-metricity of the p-adic norm. The result means that stepwise analytic continuation is not possible and one can construct square root function

only in the union of p-adic convergence spheres associated with the points $x(k, s) = sp^k$ which correspond to ordinary p-adic numbers.

d) By the scaling properties of the square root function the convergence radius $R(x(k, s)) \equiv R(k)$ is related to $R(x(0, s)) \equiv R(0)$ by the scaling factor p^{-k}:

$$R(k) = p^{-k}R(0) , \qquad (38)$$

so that the convergence sphere expands as a function of the p-adic time coordinate. The study of the convergence reduces to the study of the series at points $x = s = 1, ..., k - 1$ with a unit p-adic norm.

e) Two neighboring points $x = s$ and $x = s+1$ cannot belong to the same convergence sphere: this would lead to a contradiction with the basic results of about square root function at integer points. Therefore the convergence radius satisfies the condition

$$R(0) < 1 . \qquad (39)$$

The requirement that the convergence is achieved at all points of the real axis implies

$$R(0) = \frac{1}{p} ,$$
$$R(p^k s) = \frac{1}{p^{k+1}} . \qquad (40)$$

If the convergence radius is indeed this, then the region, where the square root is defined, corresponds to a connected light cone like region defined by the condition $N_p(Im(Z)) = N_p(Re(Z))$ and $p > 2$-adic space time is the p-adic analog of the M^4 lightcone. If the convergence radius is smaller, the convergence region reduces to a union of disjoint p-adic spheres with increasing radii.

How the p-adic light cone differs from the ordinary light cone can be seen by studying the explicit form of the p-adic norm for $p > 2$ square root allowing extension $Z = x + iy + \sqrt{p}(u + iv)$

$$N_p(Z) = (N_p(det(Z)))^{\frac{1}{4}} ,$$
$$= (N_p((x^2 + y^2)^2 + 2p^2((xv - yu)^2 + (xu - yv)^2) + p^4(u^2 + v^2)^2))^{\frac{1}{4}} ,$$
$$\qquad (41)$$

where $det(Z)$ is the determinant of the linear map defined by a multiplication with Z. The definition of the convergence sphere for $x = s$ reduces to

$$N_p(det(Z_3)) = N_p(y^4 + 2p^2y^2(u^2 + v^2) + p^4(u^2 + v^2)^2)) < 1 \ . \quad (42)$$

For physically interesting case $p \ mod \ 4 = 3$ the points (y, u, v) satisfying the conditions

$$
\begin{aligned}
N_p(y) &\leq \frac{1}{p} \ , \\
N_p(u) &\leq 1 \ , \\
N_p(v) &\leq 1 \ ,
\end{aligned}
\quad (43)
$$

belong to the sphere of convergence: it is essential that for all u and v satisfying the conditions one has also $N_p(u^2 + v^2) \leq 1$. By the canonical correspondence between p-adic and real numbers, the real counterpart of the sphere $r = t$ is now the parallelpiped $0 \leq y < 1, 0 \leq u < p, 0 \leq v < p$, which expands with an average velocity of light in discrete steps at times $t = p^k$.

7.3 Convergence radius for square root function

In the following it will be shown that the convergence radius of $\sqrt{t + Z}$ is indeed non-vanishing for $p > 2$. The expression for the Taylor series of $\sqrt{t + Z}$ reads as

$$
\begin{aligned}
\sqrt{t + Z} &= = \sqrt{x} \sum_n a_n \ , \\
a_n &= (-1)^n \frac{(2n - 3)!!}{2^n n!} x^n \ , \\
x &= \frac{Z}{t} \ .
\end{aligned}
\quad (44)
$$

The necessary criterion for the convergence is that the terms of the power series approach to zero at the limit $n \rightarrow \infty$. The p-adic norm of the n:th term is for $p > 2$ given by

$$N_p(a_n) = N_p(\frac{(2n - 3)!!}{n!})N_p(x^n) < N_p(x^n)N_p(\frac{1}{n!}) \ . \quad (45)$$

60

The dangerous term is clearly the $n!$ in the denominator. In the following it will be shown that the condition

$$U \equiv \frac{N_p(x^n)}{N_p(n!)} < 1 \; for \; N_p(x) < 1 \; , \tag{46}$$

holds true. The strategy is as follows:
a) The norm of x^n can be calculated trivially: $N_p(x^n) = p^{-Kn}, K \geq 1$.
b) $N_p(n!)$ is calculated and an upper bound for U is derived at the limit of large n.

7.3.1 p-Adic norm of $n!$ for $p > 2$

Lemma 1: Let $n = \sum_{i=0}^{k} n(i)p^i$, $0 \leq n(i) < p$ be the p-adic expansion of n. Then $N_p(n!)$ can be expressed in the form

$$
\begin{aligned}
N_p(n!) &= \prod_{i=1}^{k} N(i)^{n(i)} \; , \\
N(1) &= \frac{1}{p} \; , \\
N(i+1) &= N(i)^{p-1} p^{-i} \; .
\end{aligned} \tag{47}
$$

An explicit expression for $N(i)$ reads as

$$N(i) = p^{-\sum_{m=0}^{i} m(p-1)^{i-m}} \; . \tag{48}$$

Proof: $n!$ can be written as a product

$$
\begin{aligned}
N_p(n!) &= \prod_{i=1}^{k} X(i, n(i)) \; , \\
X(k, n(k)) &= N_p((n(k)p^k)!) \; , \\
X(k-1, n(k-1)) &= N_p\left(\prod_{i=1}^{n(k-1)p^{k-1}} (n(k)p^k + i) \right) = N_p((n(k-1)p^{k-1})!) \; , \\
X(k-2, n(k-2)) &= N_p\left(\prod_{i=1}^{n(k-2)p^{k-2}} (n(k)p^k + n(k-1)p^{k-1} + i) \; , \right) \\
&= N_p((n(k-2)p^{k-2})!) \; , \\
X(k-i, n(k-i)) &= N_p((n(k-i)p^{k-i})!) \; .
\end{aligned} \tag{49}
$$

The factors $X(k, n(k))$ reduce in turn to the form

$$X(k, n(k)) = \prod_{i=1}^{n(k)} Y(i, k) ,$$

$$Y(i, k) = \prod_{m=1}^{p^k} N_p(ip^k + m) . \tag{50}$$

The factors $Y(i, k)$ in turn are indentical and one has

$$X(k, n(k)) = X(k)^{n(k)} ,$$
$$X(k) = N_p(p^k!) . \tag{51}$$

The recursion formula for the factors $X(k)$ can be derived by writing explicitely the expression of $N_p(p^k!)$ for a few lowest values of k:
1) $X(1) = N_p(p!) = p^{-1}$.
2) $X(2) = N_p(p^2!) = X(1)^{p-1}p^{-2}$ ($p^2!$ decomposes to $p-1$ products having same norm as $p!$ plus the last term equal to p^2.
i) $X(i) = X(i-1)^{p-1}p^{-i}$

Using the recursion formula repeatedly the explicit form of $X(i)$ can be derived easily. Combining the results one obtains for $N_p(n!)$ the expression

$$N_p(n!) = p^{-\sum_{i=0}^{k} n(i)A(i)} ,$$

$$A(i) = \sum_{m=1}^{i} m(p-1)^{i-m} . \tag{52}$$

The sum $A(i)$ appearing in the exponent as the coefficient of $n(i)$ can be calculated by using geometric series

$$A(i) = (\frac{p-1}{p-2})^2(p-1)^{i-1}(1 + \frac{i}{(p-1)^{i+1}} - \frac{(i+1)}{(p-1)^i}) ,$$

$$\leq (\frac{p-1}{p-2})^2(p-1)^{i-1} . \tag{53}$$

7.3.2   Upper bound for $N_p(\frac{x^n}{n!})$ for $p > 2$

By using the expressions $n = \sum_i n(i)p^i$, $N_p(x^n) = p^{-Kn}$ and the expression of $N_p n!$ as well as the upper bound

$$A(i) \leq (\frac{p-1}{p-2})^2(p-1)^{i-1} \ . \tag{54}$$

For $A(i)$ one obtains the upper bound

$$N_p(\frac{x^n}{n!}) \leq p^{-\sum_{i=0}^{k} n(i)p^i(K-(\frac{(p-1)}{(p-2)})^2(\frac{(p-1)}{p})^{i-1})} \ . \tag{55}$$

It is clear that for $N_p(x) < 1$ that is $K \geq 1$ the upper bound goes to zero. For $p > 3$ exponents are negative for all values of i: for $p = 3$ some lowest exponents have wrong sign but this does not spoil the convergence. The convergence of the series is also obvious since the real valued series $\frac{1}{1-\sqrt{N_p(x)}}$ serves as a majorant.

7.4 $p = 2$ case

In $p = 2$ case the norm of a general term in the series of the square root function can be calculated easily using the previous result for the norm of $n!$:

$$N_p(a_n) \quad = \quad N_p(\frac{(2n-3)!!}{2^n n!})N_p(x^n) = 2^{-(K-1)n+\sum_{i=1}^{k} n(i)\frac{i(i+1)}{2^{i+1}}} \ . \tag{56}$$

At the limit $n \to \infty$ the sum term appearing in the exponent approaches zero and convergence condition gives $K > 1$, so that one has

$$N_p(Z) \quad \equiv \quad (N_p(det(Z)))^{\frac{1}{8}} \leq \frac{1}{4} \ . \tag{57}$$

The result does not imply disconnected set of convergence for square root function since the square root for half odd integers exists:

$$\sqrt{s + \frac{1}{2}} = \frac{\sqrt{2s+1}}{\sqrt{2}} \ , \tag{58}$$

so that one can develop square as a series in all half odd integer points of the p-adic axis (points which are ordinary p-adic numbers). As a consequence, the structure for the set of convergence is just the 8-dimensional counterpart of the p-adic light cone. Space-time has natural binary structure in the sense that each $N_p(t) = 2^k$ cylinder consists of two identical p-adic 8-balls (parallelpipeds as real spaces).

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